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PROCEEDINGS

OF THE

LONDON MATHEMATICAL SOCIETY.

VOL. XXIII.

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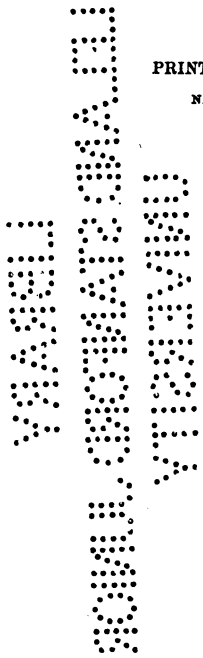

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PROCEEDINGS

OF THE

LONDON MATHEMATICAL SOCIETY.

VOL. XXIII.

TWENTY-EIGHTH SESSION, 1891–92.

November 12th, 1891.

ANNUAL GENERAL MEETING, held at 22 Albemarle Street, W.

Prof. GREENHILL, F.R.S., President, in the Chair.

The President announced the recent decease of Mr. H. M. Jeffery, F.R.S., who was elected a member January 14th, 1875.

The Treasurer (Mr. A. B. Kempe) read his Report. Its reception was moved by Mr. S. Roberts, seconded by Lt.-Col. Cunningham, R.E., and carried unanimously.

At the request of the Chairman, Mr. G. Heppel consented to act as Auditor.

From the Report of the Secretaries, it appeared that the number of the members during the Session had been 208, now reduced to 200, by the deaths of four members, and retirement of four others. The number of compounders is 87.

The Society had to regret the loss by death of Sir Robert Nicholas Fowler, Bart., M.P., who was elected a member March 19th, 1866; of Dr. John Casey, F.R.S., who was elected a member November 12th, 1874; of Mr. Henry Martyn Jeffery, F.R.S., who was elected a member January 14th, 1875; and of Mr. W. J. Curran Sharp, wh was elected a member December 11th, 1879.

The third presentation of the De Morgan Medal was made at the annual meeting, November 13th, 1890, to Lord Rayleigh.

The following communications had been made or received :—

The Presidential Address: On the Influence of Applied on the Progress of Pure Mathematics: Mr. J. J. Walker.

Proofs of Steiner's Theorems relating to Circumscribed and Inscribed Conics: Prof. G. B. Mathews.

On an Algebraical Integral of Two Differential Equations: Mr. R. A. Roberts.

Some Theorems in Elementary Geometry: Mr. Oscher Ber.

On the Analytical Representation of Heptagrams: Mr. L. J. Rogers.

On the Stability of a Plane Plate under Thrusts in its own Plane, with applications to the "Buckling" of the Sides of a Ship: Mr. G. H. Bryan.

On the Application to Matrices of any Order of the Quaternion Symbols S and V: Dr. Henry Taber.

On the Reversion of Partial Differential Expressions with two Independent and two Dependent Variables: Mr. E. B. Elliott.

On Newton's Classification of Cubic Curves: Mr. W. W. Rouse Ball.

On the q-series derived from the Elliptic and Zeta Functions of $\frac{1}{3}K$ and $\frac{1}{4}K$: Dr. J. W. L. Glaisher.

On a certain Class of Plane Quartics: Prof. G. B. Mathews.

Two Notes on Isoscelians: Mr. R. Tucker.

The Equation of the Bitangential of the Quintic: Mr. W. E. Heal.

The Oscillations of a Spheroid in a Viscous Liquid: Mr. J. Buchanan.

On Node- and Cusp-Loci, which are also Envelopes: Prof. M. J. M. Hill.

On the Partitions of a Polygon: Prof. Cayley.

Wave-Motion in a Hetereogeneous Heavy Fluid: Mr. A. E. H. Love.

On the Disturbance produced by an Element of a Plane Wave of Sound or Light: Mr. A. B. Basset.

Yoke-Chains and Multipartite Compositions in connexion with the Analytical Forms called "Trees": Major P. A. MacMahon.

On Functions determined from their Discontinuities, and a certain Form of Boundary Condition: Prof. W. Burnside.

Relations between the Divisors of the First n Numbers: Dr. J. W. L. Glaisher.

On a certain Riemann's Surface: Prof. W. Burnside.

Quartic Equations interpreted by the Parabola: Mr. G. Heppel.

Systems of Spherical Harmonics: Mr. E. W. Hobson.

On certain Properties of Symmetric, Skew Symmetric, and Orthogonal Matrices: Dr. H. Taber.

An application of the Method of Images to the Conduction of Heat: Mr. G. H. Bryan.

A Property of the Circum-Circle: Mr. R. Tucker.

Spherical Harmonics of Fractional Order: Mr. R. A. Sampson.

Steiner's Poristic Systems of Spheres: Prof. G. B. Mathews.

Geometrical Metamorphoses by Partition and Transformation: Mr. H. Perigal.

Some Theorems concerning Groups of Totitives of n: Prof. H. W. Lloyd Tanner.

Compounded Solutions in the Calculus of Variations: Mr. E. P. Culverwell.

The paper on "Stability of Orbits," by Prof. Greenhill (read May 10th, 1888) was received during the Session, and has been printed in the *Proceedings* (Vol. XXII., pp. 264–305).

The same Journals had been subscribed for and the same exchanges made as in the preceding Session.

The meeting next proceeded to the election of the new Council. The scrutators (Professor W. H. H. Hudson and Mr. R. A. Sampson) having examined the balloting lists, declared the following gentlemen duly elected:—

President, Professor Greenhill, F.R.S.; Vice-Presidents, Dr. J. Larmor, Major MacMahon, R.A., F.R.S., Mr. J. J. Walker, F.R.S.; Treasurer, Mr. A. B. Kempe, F.R.S.; Secretaries, Messrs. M. Jenkins and R. Tucker. Other Members: Messrs. A. B. Basset, F.R.S., E. B. Elliott, F.R.S., Dr. Forsyth, F.R.S., Dr. Glaisher, F.R.S., Mr. J. Hammond, Dr. M. J. M. Hill, Messrs. C. Leudesdorf, A. E. H. Love and S. Roberts, F.R.S.

The Chairman announced that Mr. Sampson had kindly drawn up a Catalogue of the Books in the Library. A vote of thanks to Mr. Sampson was unanimously carried by acclamation.

The following communications were made:—

On Selective and Metallic Reflection: Mr. Basset.

The Contacts of Systems of Circles: Mr. A. Larmor. (Communicated by Dr. J. Larmor.)

On a Class of Automorphic Functions: Professor W. Burnside. (Communicated by the President.)

Note on the Identity $4(x^p-1)/(x-1) = Y^2 \pm pZ^2$: Professor G. B. Mathews. (Communicated by the President.)

On the Classification of Binodal Quartic Curves: The late Mr. H. M. Jeffery.

Note on finding the G points of a given Circle with respect to a given Triangle of reference: Mr. J. Griffiths.

Researches in the Calculus of Variations—(1) Discriminating Conditions in Isoperimetrical Problems: Mr. E. P. Culverwell.

Note on Clifford's paper "On Syzygetic Relations among the powers of Linear Quartics": Professor Cayley. (The last four papers were communicated by Mr. Tucker.)*

* Mr. Basset suggested that authors should furnish abstracts of their papers suitable for publication in *Nature*. These abstracts, he believed, would materially increase the interest which the outside public, as well as the members of the Society, take in the printed record of *Proceedings*, and so would advance the interests of the Society. The meeting expressed their concurrence with his remarks. (*Cf.* a wish expressed by Professor Sylvester at the Society's meeting, February 10th, 1887.)

The second hypothesis is, that the potential energy of the system contains a term, which depends upon the relative displacements of ether and matter. This portion of the potential energy, which we shall denote by W_3, represents the mutual action of ether upon matter; and if we suppose that the corresponding forces are linear functions of the relative displacements, W_3 will be of the form

$$W_3 = \tfrac{1}{2}\{A(u-u_1)^2 + B(v-v_1)^2 + C(w-w_1)^2 + 2A'(v-v_1)(w-w_1) + 2B'(w-w_1)(u-u_1) + 2C'(u-u_1)(v-v_1)\} \quad \ldots\ldots(3).$$

The potential energy W of the system accordingly consists of three parts, and therefore

$$W = W_1 + W_2 + W_3 \quad \ldots\ldots(4).$$

If T_1, ρ_0 be the kinetic energy and density of the ether, and T_2, ρ_1 of the matter,

$$\left.\begin{aligned} T_1 &= \tfrac{1}{2}\rho_0(\dot{u}^2+\dot{v}^2+\dot{w}^2) \\ T_2 &= \tfrac{1}{2}\rho_1(\dot{u}_1^2+\dot{v}_1^2+\dot{w}_1^2) \end{aligned}\right\} \quad \ldots\ldots(5).$$

The third hypothesis is, that the kinetic energy of the system contains a term, due to the mutual action of ether upon matter, whose effect is to cause the ether to behave as if its density were æolotropic. This is the hypothesis introduced by Rankine and Lord Rayleigh;* and by employing Sir W. Thomson's subsidiary hypothesis,† that the ether behaves like an elastic solid, whose resistance to compression is a negative quantity, whose numerical value is slightly less than four-thirds of its rigidity, we can deduce Fresnel's formulæ for the intensity of light reflected at the surface of a transparent isotropic medium, and also obtain a satisfactory theory of double refraction.‡ Accordingly, if T_3 denote this term,

$$T_3 = \tfrac{1}{2}(P\dot{u}^2 + Q\dot{v}^2 + R\dot{w}^2 + 2P'\dot{v}\dot{w} + 2Q'\dot{w}\dot{u} + 2R'\dot{u}\dot{v}) \quad \ldots\ldots(6).$$

It thus appears that we have introduced eighteen additional constants into our equations. We may, however, get rid of three of them by a proper choice of axes, leaving fifteen, which ought to be sufficient to construct a theory capable of explaining most optical phenomena.

---

* Hon. J. W. Strutt, *Phil. Mag.*, June, 1871.
† *Phil. Mag.*, (5) Vol. xxvi., p. 414.
‡ Glazebrook, *ibid.*, p. 521.

If we suppose the axes chosen so as to get rid of the product terms in T_2, and write $T_1' = T_1 + T_3$, we may put

$$T_1' = \tfrac{1}{2}(P\dot{u}^2 + Q\dot{v}^2 + R\dot{w}^2),$$

since ρ_0 may be supposed to be merged in P, Q, R.

3. The equations of motion of the system may now be deduced by the Principle of Least Action, viz.,

$$\iiiint \delta(T_1' + T_2 - W_1 - W_2 - W_3)\, dx\, dy\, dz\, dt = 0,$$

and will be found to be

$$P\frac{d^2u}{dt^2} = m\frac{d\Delta}{dx} + n\nabla^2 u - \frac{dW_3}{du} \quad \ldots\ldots\ldots\ldots\ldots\ldots (7),$$

with two similar equations, and

$$\rho_1\frac{d^2u_1}{dt^2} = -\frac{dW_3}{du_1} - \frac{dW_2}{du_1} \quad \ldots\ldots\ldots\ldots\ldots\ldots (8),$$

with two similar equations.

We shall now confine our attention to isotropic media. In this case

$$P = Q = R = \rho,$$
$$A = B = C = a^2, \quad A' = B' = C' = 0,$$
$$\mathfrak{A} = \mathfrak{B} = \mathfrak{C} = \beta^2, \quad \mathfrak{A}' = \mathfrak{B}' = \mathfrak{C}' = 0,$$

and the equations of motion accordingly become

$$\rho\frac{d^2u}{dt^2} = m\frac{d\Delta}{dx} + n\nabla^2 u - a^2(u - u_1) \quad \ldots\ldots\ldots\ldots\ldots\ldots (9),$$

$$\rho_1\frac{d^2u_1}{dt^2} = a^2(u - u_1) - \beta^2 u_1 \quad \ldots\ldots\ldots\ldots\ldots\ldots (10),$$

with two similar equations.

4. Von Helmholtz has introduced a viscous term into the equations of motion of the matter, which has been objected to by Sir W. Thomson.* I think, however, that this term may be justified on two independent grounds. It is an experimental fact, that when vibrations are excited in any material substance, the vibrations are

* *Lectures on Molecular Dynamics*, p. 149.

gradually damped by internal friction, and the motion ultimately dies away. This state of things may be represented mathematically by the introduction of a viscous term into the equations of motion of the matter. If, however, we do not wish to introduce the hypothesis of internal friction into our equations, the viscous term may still be accounted for. When a molecule of matter is thrown into vibration by ethereal waves, the physical characteristics of the motion will be much the same as the motion of a sphere attached to a spring and vibrating in air under the influence of impinging waves of sound;* consequently the vibrations of the molecules will generate secondary waves in the ether, which will travel away into space, carrying with them the energy which is communicated by the impinging waves, and the equations of motion of the matter will therefore contain a viscous term. In the place of (10), we shall therefore assume that the equations of motion of the matter are of the form

$$\rho_1 \frac{d^2u_1}{dt^2} + 2h\frac{du_1}{dt} = a^2(u-u_1) - \beta^2 u_1 \quad \ldots\ldots\ldots\ldots\ldots(11).$$

5. Let us now consider the propagation of waves of light, which are travelling through the medium in the positive direction of the axis of z, and let the displacement be parallel to the axis of x.

Equation (9) now becomes

$$\rho\frac{d^2u}{dt^2} = n\frac{d^2u}{dz^2} + a^2(u_1-u) \quad \ldots\ldots\ldots\ldots\ldots\ldots(12).$$

Equations (11) and (12) are the same as those upon which von Helmholtz bases his theory of anomalous dispersion, but I wish to conduct the integration in a different manner, in order to bring in the free period of the matter vibrations.

Assume
$$u = Ae^{-qz+2i\pi/\tau\cdot(z/V-t)},$$
$$u_1 = A_1 e^{-qz+2i\pi/\tau\cdot(z/V-t)};$$

then (12) and (11) become

$$\left\{\frac{4\pi^2\rho}{\tau^2} + n\left(q-\frac{2i\pi}{V\tau}\right)^2 - a^2\right\}A = -a^2A_1 \quad \ldots\ldots\ldots\ldots(13),$$

$$\left\{\frac{4\pi^2\rho_1}{\tau^2} - \beta^2 - a^2 + \frac{4i\pi h}{\tau}\right\}A_1 = -a^2A \quad \ldots\ldots\ldots\ldots(14).$$

* Lord Rayleigh, *Theory of Sound*, Vol. II., § 334; Basset, *Proc. Lond. Math. Soc.*, Vol. XXII., pp. 328, 329.

The free period κ of the matter vibrations is obtained by putting $a = 0$ in (11), and is determined by the equation

$$\beta^2 = \frac{4\pi^2\rho_1}{\kappa^2} + \frac{h^2}{\rho_1} \quad \text{.....(14a)};$$

whence (14) becomes

$$\frac{A_1}{A} = -\frac{a^2}{4\pi^2\rho_1(\tau^{-2} - \kappa^{-2}) - a^2 - h^2/\rho_1 + 4\iota\pi h/\tau} \quad \text{.....(15).}$$

From the problem of the sphere vibrating in air, we infer that the amplitudes of the vibrations of the molecules of matter are very much smaller than those of the ether; accordingly, a^2 must be very small compared with ρ_1, the density of the matter. If, however,

$$\frac{1}{\tau^2} = \frac{1}{\kappa^2} + \frac{a^2}{4\pi^2\rho_1},$$

in which case the period of the waves would be sensibly equal to that of the matter, A_1 would be infinite in the absence of friction. This conclusion is contradicted by experience, and necessitates the introduction of the viscous term.

Let D denote the real part of the denominator of (15); then, if we eliminate A, A_1 from (13) and (14), and equate the real and imaginary parts, we shall obtain

$$\left(\frac{4\pi^2\rho}{\tau^2} + nq^2 - \frac{4\pi^2 n}{V^2\tau^2} - a^2\right)D + \frac{16\pi^2 nhq}{V\tau^2} = a^4 \quad \text{.....(16),}$$

$$\left(\frac{4\pi^2\rho}{\tau^2} + nq^2 - \frac{4\pi^2 n}{V^2\tau^2} - a^2\right)h - \frac{nqD}{V} = 0 \quad \text{.....(17).}$$

Now q is the coefficient of absorption of the medium; and, since the absorption is very slight in most transparent media, it follows that q, and therefore h, are very small.

Let P denote the coefficient of h in (17); then, eliminating h between (16) and (17), we obtain

$$P^2D - a^4P + \frac{16\pi^2 n^2 q^2 D}{V^2\tau^2} = 0;$$

whence

$$P = \tfrac{1}{2}\left\{a^4 + \left(a^8 - \frac{64\pi^2 n^2 q^2 D^2}{V^2\tau^2}\right)^{\frac{1}{2}}\right\} \Big/ D,$$

the positive sign being taken, because $P = a^4/D$ when $q = 0$.

Expanding the quantity under the radical, we obtain

$$P = \frac{\alpha^4}{D} - \frac{16\pi^2 n^2 q^2 D}{\alpha^4 V^2 \tau^2} \quad \text{.....(18).}$$

Let μ be the index of refraction; then $n/V^2 = \rho_0 \mu^2$, where ρ_0 is the density of the ether in vacuo; whence (18) becomes

$$\mu^2 = \left(1 - \frac{4nq^2}{\alpha^4}\right)^{-1} \left\{ \frac{\rho}{\rho_0} + \frac{nq^2\tau^2}{4\pi^2\rho_0} - \frac{\alpha^2\tau^2}{4\pi^2\rho_0} \left[1 + \frac{\alpha^2}{4\pi^2\rho_1(\tau^{-2} - \kappa^{-2}) - \alpha^2 - h^2/\rho_1} \right] \right\}$$

.....(19).

If we suppose that the absorbing power of the medium is so small that q and h may be neglected, we obtain the more simple formula

$$\mu^2 = \frac{\rho}{\rho_0} - \frac{\alpha^2\tau^2}{4\pi^2\rho_0} \left\{ 1 + \frac{\alpha^2\kappa^2\tau^2}{4\pi^2\rho_1(\kappa^2 - \tau^2) - \alpha^2\kappa^2\tau^2} \right\} \quad \text{.....(20);}$$

but it is worthy of note that, even when q and h are not neglected, it is possible for μ^2 to become a negative quantity.

6. From (20) we obtain

$$\frac{d\mu^2}{d(\tau^2)} = -\frac{\alpha^2 \left\{ (\kappa^2 - \tau^2)^2 + \alpha^2\kappa^2\tau^4/4\pi^2\rho_1 \right\}}{4\pi^2\rho_0(\kappa^2 - \tau^2 - \alpha^2\kappa^2\tau^2/4\pi^2\rho_1)},$$

which shows that $d\mu^2/d\tau^2$ is a negative quantity; *accordingly* μ *decreases as* τ *increases.*

Since ρ is the density of the ether when loaded with matter, $\rho > \rho_0$; hence, when $\tau = 0$, $\mu > 1$. As τ increases, μ^2 diminishes to unity, it then becomes less than unity, and then vanishes and changes sign. At this point an absorption band commences, and continues until $\tau^2 = \kappa^2(1 + \alpha^2/4\pi\rho_1)^{-2}$, when $\mu^2 = -\infty$. As soon as τ exceeds this critical value (which is nearly equal to the free period of the matter vibrations), μ^2 becomes a large positive quantity; and as τ continues to increase, μ^2 proceeds to diminish, and again vanishes and changes sign, after which it always remains negative.

Equation (20) accordingly represents a substance, which is absolutely opaque to all waves whose periods exceed a certain value τ_1; it is transparent to waves whose periods lie between τ_1 and a quantity which is very nearly equal to κ, the free period of the matter; it is opaque to waves whose periods lie between κ and a quantity τ_2, which is less than κ and which corresponds to $\mu = 0$; and transparent to waves whose periods are less than τ_2.

The dispersion above and below the absorption band, which lies between κ and τ_2, is *anomalous*, since the value of μ, when τ is slightly greater than κ, is greater than when τ is slightly less than τ_2. If we suppose that κ corresponds to the line D, whilst τ_2 corresponds to F, there will be an absorption band in the green, and the order of colours in the visible portion of the spectrum will be blue, indigo, violet, then an absorption band in the green, followed by red, orange, yellow.

7. We must now consider the reflection of light at the surface of a medium which exhibits anomalous dispersion, and we shall suppose that the absorption band lies between D and F.

In forming the equations of motion by means of the Principle of Least Action, we see that there are no surface integral terms which arise from W_2 and W_3; it therefore follows, that the boundary conditions at the surface of separation of two different media, are unaffected by the presence of the terms depending on the action of the matter. These conditions will therefore be, continuity of the displacements and stresses arising from the action of the ether. With regard to the physical properties of the ether, I shall provisionally adopt the hypothesis of Sir W. Thomson, that the latter is to be treated as an elastic medium, whose resistance to compression is a negative quantity, whose numerical value is slightly less than four-thirds of the rigidity.

Under these circumstances, the intensities of the reflected and refracted light are given by Fresnel's formulæ; and so long as $\mu > 1$ the reflection takes place in the same manner as from glass. If, however, the incident light is white, and κ lies within the visible portion of the spectrum, say between D and F, it follows that for certain rays of the spectrum $\mu < 1$, in which case there will be a critical angle. Hence those rays for which $\mu < \sin i$, will be totally reflected with a change of phase; and, since the change of phase is different, according as the incident light is polarized in or perpendicularly to the plane of incidence, it follows that, when the plane of polarization is neither parallel nor perpendicular to the plane of incidence, the reflected light will be elliptically polarized.

If e, e_1 be the changes of phase, when $\mu < 1$, corresponding to light polarized in and perpendicularly to the plane of incidence,

$$\left.\begin{aligned} \tan\frac{\pi e}{\lambda} &= (\tan^2 i - \mu^2 \sec^2 i)^{\frac{1}{2}} \\ \tan\frac{\pi e_1}{\lambda} &= \mu^{-2}(\tan^2 i - \mu^2 \sec^2 i)^{\frac{1}{2}} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (21).$$

There will also be another set of rays, for which μ^2 is a negative quantity, and we shall now show that these rays will also be totally reflected with a change of phase.

Let $\mu^2 = -\nu^2$; then

$$\sin i = \iota\nu \sin r, \quad \cos r = \nu^{-1}(\nu^2 + \sin^2 i)^{\frac{1}{2}}.$$

Now, if A' be the amplitude of the reflected light, when the incident light is polarized *in* the plane of incidence,

$$\frac{A'}{A} = -\frac{\sin(i-r)}{\sin(i+r)}$$

$$= \frac{\cos i - \iota(\nu^2 + \sin^2 i)^{\frac{1}{2}}}{\cos i + \iota(\nu^2 + \sin^2 i)^{\frac{1}{2}}}$$

$$= e^{-2\iota\pi f/\lambda},$$

where $$\tan \pi f/\lambda = (\nu^2 + \sin^2 i)^{\frac{1}{2}}/\cos i \quad \ldots\ldots\ldots\ldots\ldots\ldots (22),$$

which shows that the reflection is total, and is accompanied by a change of phase whose value is determined by (22).

When the light is polarized perpendicularly to the plane of incidence,

$$\frac{B'}{B} = \frac{\tan(i-r)}{\tan(i+r)}$$

$$= e^{2\iota\pi f_1/\lambda},$$

where $$\tan\frac{\pi f_1}{\lambda} = \frac{(\nu^2 + \sin^2 i)^{\frac{1}{2}}}{\nu^2 \cos i} \quad \ldots\ldots\ldots\ldots\ldots\ldots (23),$$

which shows that the reflection is total and is accompanied by a change of phase.

From (22) and (23) we see, that the change of phase is different according as the incident light is polarized in or perpendicularly to the plane of incidence; whence in this case also it follows, that the reflected light is elliptically polarized.

8. We must now enquire how far these results will explain the selective reflection which is produced by substances which exhibit anomalous dispersion.

We shall suppose that there is a single absorption band, which begins at D, corresponding to $\mu^2 = -\infty$, $\tau = \kappa$, and ends at F, corresponding to $\mu = 0$, $\tau = \tau_2$; hence the substance will be opaque to

light whose periods lie between these limits, and will totally reflect it. Let τ_3 be a quantity less than τ_2, and such that the corresponding value of μ is equal to $\sin i$, where i is the angle of incidence. Then all the rays whose periods lie between τ_3 and τ_2 will be totally reflected with a change of phase; whilst light whose periods are less than τ_3, or which lie between τ_1 and κ, will be reflected in the ordinary manner. If, therefore, sunlight be incident upon the substance, the rays which lie between D and a line a little beyond F, will be brilliantly reflected, and the reflected light will be of a greenish tint. The transmitted light, on the other hand, will be of a reddish blue colour, owing to the absence of the green rays.

9. Let us next suppose, that the reflected light is passed through a Nicol's prism, whose principal section is parallel to the plane of incidence; then the component vibrations perpendicular to the plane of incidence cannot get through the Nicol, and we have therefore only to deal with the vibrations in the plane of incidence.

Now, if the angle of incidence is equal to the polarizing angle of any of the rays which are reflected in the ordinary manner, these rays will be absent; and consequently the colour of the reflected light, when viewed through the Nicol, will be different from what it is when viewed by the naked eye. If, for example, the absorption band commenced a little above D and ended at F, the reflected light would be green; but if the angle of incidence is equal to the polarizing angle of the D rays, these rays will be extinguished by the Nicol, and the reflected light will appear to be of a peacock blue, owing to the absence of yellow rays.

The results of the last two sections, explain the selective reflection produced by fuchsine.

Metallic Reflection.

10. We shall now enquire how far von Helmholtz' theory is capable of explaining metallic reflection.

Let the metallic reflector be in contact with a transparent medium such as air; let the axis of x be normal to the reflecting surface, and let the axis of z be parallel to the traces of the waves. Then the incident, reflected, and refracted waves may be written

$$A\epsilon^{2i\pi/V\tau.(lx+my-Vt)},$$

$$A'\epsilon^{2i\pi/V\tau.(l'x+my-Vt)},$$

$$A_1\epsilon^{2i\pi/V_1\tau.(l_1x+m_1y-V_1t)},$$

where $$l = -l' = -\cos i, \quad l_1 = -\cos r,$$

$$m = \sin i, \qquad m_1 = \sin r.$$

Since the coefficients of y must be the same in all three waves, we obtain

$$\frac{\sin i}{\sin r} = \frac{V}{V_1} = \mu, \text{ say } \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (24).$$

If the medium were transparent, V_1 would be a real quantity, and would represent the velocity of propagation, and μ would be the index of refraction; but, since there is no regularly refracted wave when the second medium is a metal, V_1 cannot be a real quantity. We must therefore assume that V_1 is a complex quantity, in which case r and μ will also be complex.

For these reasons it is often said that the index of refraction of metals is a complex quantity. The expression is not, however, very happily chosen, since there is no such thing as an index of refraction in the case of metals. If, however, we regard the index of refraction for metals as a convenient name for the mathematical quantity which is defined by (24), there will be no danger of any ambiguity.

When the incidence is normal, we shall have $l = l_1 = -1$; if therefore we put*

$$\mu = Re^{i\theta},$$

we shall have $$V_1 = VR^{-1}e^{-i\theta},$$

and the refracted wave becomes

$$A_1 \exp\left\{\frac{2\pi R}{V\tau}x\cos\theta - \frac{2i\pi}{\tau}\left(\frac{Rx}{V}\cos\theta + t\right)\right\};$$

the real part of which (supposing that A_1 is real) is

$$A_1 e^{(2\pi R/V\tau)x\sin\theta}\cos\frac{2\pi}{V\tau}(Rx\cos\theta + Vt),$$

corresponding to the incident wave

$$A\cos\frac{2\pi}{V\tau}(x + Vt).$$

Since x is negative in the second medium, it follows that θ must be positive; and therefore μ is a complex quantity, whose imaginary part is positive.

* θ is the quantity which Lord Rayleigh has denoted by $-\alpha$.

Hence θ may have any value lying between 0 and π; and, since

$$\mu^2 = R^2 (\cos 2\theta + \iota \sin 2\theta),$$

it follows that the real part of μ^2 may be either positive or negative.

11. Cauchy's formulæ for metallic reflection may be obtained by eliminating r from Fresnel's formulæ by means of (24), and then transforming them by assuming that μ is a complex quantity.* This transformation introduces two new quantities R and θ, which can be determined by experiment. Now the experiments of Jamin,† and the calculations of Eisenlohr, both show that for a variety of metals $\theta > \frac{1}{4}\pi$, and consequently, for such metals, the real part of μ^2 must be a *negative* quantity. Lord Rayleigh, on the other hand, has shown, that if we attempt to account for metallic reflection on the elastic solid theory,‡ by assuming that the effect of metallic reflection depends upon the existence of a viscous term; or upon the electromagnetic theory,§ by taking into account the conductivity; physical considerations require that the real part of μ^2 should be *positive*. We are therefore unable to explain metallic reflection on either of these theories.

The only other course open to us is to construct some theory, based upon the mutual reaction of ether and matter, which will furnish an expression for μ^2, which is a complex quantity whose real part is negative; and we shall now show that von Helmholtz' theory leads to such an expression.

We shall still continue to suppose that the wave is propagated in the negative direction of the axis of x, and that the direction of vibration is parallel to z, in which case the equations of motion are

$$\left.\begin{aligned} \rho \frac{d^2w}{dt^2} &= n \frac{d^2w}{dx^2} + a^2 (w_1 - w) \\ \rho_1 \frac{d^2w_1}{dt^2} + 2h \frac{dw_1}{dt} &= a^2 (w - w_1) - \beta^2 w_1 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots (25).$$

We shall find it convenient to integrate these equations in a somewhat different manner, and shall put

$$\left.\begin{aligned} w &= A \epsilon^{-2\iota\pi/\tau \cdot (x/V + t)} \\ w_1 &= A_1 \epsilon^{-2\iota\pi/\tau \cdot (x/V + t)} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (26);$$

* Eisenlohr, *Pogg. Ann.*, Vol. CIV., p. 368.
† *Ann. de Chimie et de Physique*, Vol. XIX., p. 296.
‡ *Phil. Mag.*, May, 1872.
§ *Ibid.*, August, 1881.

also μ, the pseudo-index of refraction, will be defined by the equation

$$\mu = U/V = n^{\frac{1}{2}}/V\rho_0^{\frac{1}{2}} \quad \text{.....(27),}$$

where U or $(n/\rho_0)^{\frac{1}{2}}$ is the velocity of light in vacuo. Substituting from (26) in (25), and taking account of (14a), we obtain

$$\left.\begin{aligned} &\left\{\frac{4\pi^2\rho_0}{\tau^2}\left(\frac{\rho}{\rho_0}-\mu^2\right)-a^2\right\}A = -a^2A_1 \\ &\left\{4\pi^2\rho_1\left(\frac{1}{\tau^2}-\frac{1}{\kappa^2}\right)-a^2-\frac{h^2}{\rho_1}+\frac{4\iota\pi h}{\tau}\right\}A_1 = -a^2A \end{aligned}\right\} \quad \text{... (28).}$$

Let $$D = 4\pi^2\rho_1\left(\frac{1}{\tau^2}-\frac{1}{\kappa^2}\right)-a^2-\frac{h^2}{\rho_1} \quad \text{.....(29);}$$

then, eliminating A and A_1 from (28), we obtain

$$\begin{aligned}\mu^2 &= \frac{\rho}{\rho_0}-\frac{a^2\tau^2}{4\pi^2\rho_0}\left(1+\frac{a^2}{D+4\iota\pi h/\tau}\right) \\ &= \frac{\rho}{\rho_0}-\frac{a^2\tau^2}{4\pi^2\rho_0}\left(1+\frac{a^2(D-4\iota\pi h/\tau)}{D^2+16\pi^2h^2/\tau^2}\right) \quad \text{..... (30).}\end{aligned}$$

In order to apply this result to metallic reflection, we shall suppose that h is a small quantity, whose square may be neglected; under which circumstances, the *real* part of μ^2 becomes equal to

$$\frac{\rho}{\rho_0}-\frac{a^2\tau^2}{4\pi^2\rho_0}\left\{1+\frac{a^2\kappa^2\tau^2}{4\pi^2\rho_1(\kappa^2-\tau^2)-a^2\kappa^2\tau^2}\right\} \quad \text{..... (31).}$$

From (20) we see that this expression is of the same form as the square of the refractive index of a substance, which produces anomalous dispersion, and has a single absorption band; and it may be negative in two distinct ways.

In the first place, if τ_2 be the *least* value of τ which makes (31) vanish, and τ' the value of τ which makes the denominator of the third term of (31) zero, the real part of μ^2 will be negative for all values of τ lying between τ_2 and τ'. In the second place, there is another value τ_1 of τ, which is greater than τ', which makes (31) vanish; and for all values of τ which are greater than τ_1, (31) is negative. We may therefore account for metallic reflection in two ways. In the first place, we may suppose that the period of the free vibrations of the matter is such that throughout the luminous portion of the spectrum, and some distance on either side beyond it,

τ lies between τ_2 and τ'; or, in the second place, we may suppose that throughout this range $\tau > \tau_1$. Now, metals reflect rays of dark heat in much the same way as they reflect light;* accordingly, if we adopted the first hypothesis, it would be necessary to suppose that κ, the free period of the matter vibrations, lies below the infra-red portion of the spectrum; moreover, under these circumstances, metals would be transparent to waves whose periods are greater than κ. If, on the other hand, we adopted the second hypothesis, it would be necessary to suppose that κ corresponds to a point in or above the ultra-violet portion of the spectrum; in which case metals would be transparent to waves of very short period. It is, however, necessary to point out, that the theory which has been developed, only applies to a medium having a single absorption band; whereas there is no *a priori* reason why metals should not have several. A theory such as von Helmholtz' could be developed, so as to apply to a medium having a number of absorption bands; and there can be little doubt that the real part of μ^2 would be given by an expression of much the same form as that furnished by Sir W. Thomson's molecular theory.

12. The present theory, being an elastic solid theory, does not enable us to explain the experiments of Kerr† on reflection from a magnetized iron reflector, or those of Kundt‡ on the transmission of light through thin metallic films which are normally magnetized. In order to do this, it would first be necessary to construct an electromagnetic theory of metallic reflection, by assuming that the energy of the system contains a term in which the quantities by which the motion of the matter is specified, enter into combination with electromagnetic quantities. A theory of this kind ought to lead to Cauchy's formulæ for metallic reflection, or formulæ of a very similar character; and, as soon as this has been done, an attempt might be made to explain the experiments of Kerr and Kundt by means of Hall's effect. It is by no means improbable that, when magnetic force acts upon a medium, such as iron, which is strongly susceptible to magnetic influence, the periods of the free vibrations of the matter may

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* Magnus, *Pogg. Ann.*, Vol. CXXXIX.

† *Phil. Mag.*, May, 1877, and March, 1878.

‡ *Berlin Sitzungsberichte*, June 10th, 1884; translated *Phil. Mag.* (5), Vol. XVIII., p. 308.

be altered, and may under these circumstances contain a term depending on the magnetic force.

[*February 3rd*, 1892.

Since this paper was read, a paper has been communicated to the Society by Mr. Larmor, on "Gyrostatically Loaded Media," which suggests a theory which might possibly explain Kerr's experiments.]

On the Classification of Binodal Quartic Curves. *By* HENRY M. JEFFERY, F.R.S. Received June 18th, 1891. Read November 12th, 1891.*

1. The general form of their equation, if the nodes are real, is

$$u^2(\xi x+\eta y-1) = x^2y^2+u^2(fx^2+2gxy+hy^2) \quad \ldots\ldots\ldots(A).$$

The line $u\ (\equiv cx+dy-1)$ connects the nodes, at which the pairs of tangents are x^2+hu^2, y^2+fu^2, and may be real, coincident, or imaginary. Six cases occur, according as h, f are negative, zero, or positive. It is proposed to examine all possible forms of (A), which can arise from variations of the parameters ξ, η. They are classified with reference (1) to their critical values, which yield a third double point, and (2) to those values of ξ, η, which correspond to points of undulation or folium-points.

2. Binodal quartics may be bipartite or unipartite, and have a maximum number of four inflexions.

A very instructive figure of a trinodal quartic is drawn by Plücker, and reproduced by Dr. Salmon with comments (*Higher Plane Curves*, p. 212). The two companion-figures are also drawn, the one unipartite and external, and the other quadripartite and internal, for small variations of a given parameter. Two real inflexions are seen in the trinodal, eight in each of the non-singular, quartics. Whenever two branches unite to form a node, two inflexions are lost, even if one remains at the node, as is the case when three inflexions pre-existed. Hence it follows that a binodal quartic has at most four inflexions.

* This paper, which has not been revised by the Author, is edited by the Secretaries.

Zeuthen based his conclusions, respecting the maximum number of eight inflexions, on Plücker's general form of a quartic curve,

$$xyzu+\kappa\phi^2=0,$$

in which (x, y, z, u) denote bitangents (of the first species), whose points of contact lie on the same conic (ϕ). If the quartic be non-singular, each such bitangent touches a folium, or indentation in the curve circuit, characterized by two inflexions; moreover, two points of contact may unite, and become folium-points, or points of undulation.

The reader is also referred to a series of bicrunodal quartics, with their companion-curves, which illustrates the nodal forms, the four inflexions, and the conic complex of the quartic

$$(x^2-a^2)^2+(y^2-b^2)^2=c^2$$

(*Higher Plane Curves*, p. 43).

3. The critical values will be first considered.

For a third singularity,

$$\frac{d\phi}{dx}=0, \quad \frac{d\phi}{dy}=0,$$

if ϕ denote the quartic system (A).

$$\left.\begin{aligned}\xi&=\frac{2}{u^3}xy^2-\frac{3}{u^4}cx^2y^2+\frac{2}{u}(fx+gy)-\frac{c}{u^2}(fx^2+2gxy+hy^2)\\ \eta&=\frac{2}{u^3}x^2y-\frac{3}{u^4}dx^2y^2+\frac{2}{u}(gx+hy)-\frac{d}{u^2}(fx^2+2gxy+hy^2)\end{aligned}\right\}\ldots(B),$$

$$-1=\frac{3}{u^4}x^2y^2+\frac{1}{u^2}(fx^2+2gxy+hy^2)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(C).$$

By the aid of (C), the parameters may be written

$$\left.\begin{aligned}\xi&=\frac{2}{u^3}xy^2+c+\frac{2}{u}(fx+gy)\\ \eta&=\frac{2}{u^3}x^2y+d+\frac{2}{u}(gx+hy)\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(B).$$

Every point in (C) is a singular point in some curve in the (A) system, whose parameters are these functions of its coordinates: this subsidiary curve (C) is the locus of nodes in (A).

The curve (B) may be considered as the envelop of the satellite line (ξ, η), when x, y are eliminated from (C) and (B). The

(B), when drawn, not only exhibits critical values, but *discriminates* other values of ξ, η in their vicinity, which indicate either separate branches in the companion-curves (A), or the same circuits with inflexions. See below, §§ 45, 47, for illustrations of its use in a bicuspidal system. The satellite-line of u, viz., $\xi x+\eta y-1$, has geometrical significance in bicuspidal quartics (A), when $f=h=0$.

4. A second subsidiary curve (D) is necessary to discriminate the points in (C) which denote crunodes, cusps, and acnodes in (ϕ), the (A) system. (D) is the function

$$\left(\frac{d^2\phi}{dx\,dy}\right)^2-\frac{d^2\phi}{dx^2}\cdot\frac{d^2\phi}{dy^2}.$$

By differentiating twice,

$$\frac{d^2\phi}{dx^2}=\frac{2y^2}{u^3}-\frac{12}{u^4}cxy^2+\frac{12}{u^5}c^2x^2y^2+\frac{2f}{u}-\frac{4c}{u^2}(fx+gy)+\frac{2}{u^3}c^2(fx^2+2gxy+hy^2),$$

$$\frac{d^2\phi}{dy^2}=\frac{2x^2}{u^3}-\frac{12}{u^4}dx^2y+\frac{12}{u^5}d^2x^2y^2+\frac{2h}{u}-\frac{4d}{u^2}(gx+hy)+\frac{2}{u^3}d^2(fx^2+2gxy+hy^2),$$

$$\frac{d^2\phi}{dx\,dy}=\frac{4}{u^3}xy-\frac{6xy}{u^4}(cx+dy)+\frac{12}{u^5}cdx^2y^2+\frac{2g}{u}-\frac{2}{u^2}\left[d(fx+gy)+c(gx+hy)\right]+\frac{2}{u^3}cd(fx^2+2gxy+hy^2).$$

When these values are substituted in (D), and reductions are made (most conveniently by separating the terms involving f, g, h), this function takes the simple form

$$(g^2-fh)u^4+3x^2y^2-u^2(fx^2-4gxy+hy^2) \quad \text{.....(D).}$$

Critical values in (A) will indicate crunodes, cusps, or acnodes, according as the function (D) $>, \doteq, <, 0$.

5. Accordingly, in this investigation, the subsidiary quartics (C) and (D) are drawn so that, when they intersect each other, cusps are determined in (A), and acnodes and crunodes in points on (C) inside and outside the intersections respectively. Since (C) and (D) are also generally binodal—possessing generally the same nodes as

(A), but of opposite species—they intersect in eight points besides the common nodes, and consequently there may be eight cusps in the (A) system for the same constants f, g, h.

6. If the curves (C) and (D) touch each other, the four points of quadruple contact are indicated by parameters ξ, η, which resolve four binodal quartics (A) into nodal cubics and straight lines.

In such a complex the linear factor represents a tangent at the missing node, and touches the cubic in the given point of contact. Since this contact is imaginary when the (A) system is acnodal, write the subsidiary curves with the coefficients f, h negative.

$$u^4+3x^2y^2-u^2(fx^2+2gxy+hy^2)=0 \quad \ldots\ldots\ldots\ldots (C),$$

$$(g^2-fh)u^4+3x^2y^2+u^2(fx^2-4gxy+hy^2)=0 \quad \ldots\ldots\ldots\ldots (D).$$

By addition, two conics are seen to pass through their eight intersections (apart from the eight intersections of the two nodes)

$$u^4(1+g^2-fh)-6gu^2xy+6x^2y^2=0.$$

These coincide in pairs, if $g^2=2-2fh$, when $gu^2=2xy$.

By subtraction, $$(1+fh)u^2-2(fx^2+hy^2)=0,$$

or $$(2fx-gy)(2hy-gx)=0,$$

which denote two radii vectores through the four points of contact. The curves (C) and (D) may now be written, so as to exhibit quadruple contact,

$$(gu^2-2xy)(2u^2-3gxy)+u^2(2fx-gy)(2hy-gx)=0 \ldots\ldots (C),$$

$$(gu^2-2xy)\left[(3g^2-2)u^2-3gxy\right]-u^2(2fx-gy)(2hy-gx)=0 \ldots\ldots (D).$$

For the first pair of points of contact $\left(\frac{x}{g}=\frac{y}{2f}=\frac{u}{2f^{\frac{1}{2}}}\right)$, the parameters (by § 3), are

$$\xi=c-2gf^{\frac{1}{2}}, \quad \eta=d-\frac{1}{f^{\frac{1}{2}}}(1+fh).$$

The corresponding quartic in (A) is the complex

$$u^2\left[u^2-\frac{1}{f^{\frac{1}{2}}}uy(1+fh)+hy^2\right]-2gu^2x(f^{\frac{1}{2}}u-y)+x^2(fu-y^2)=0.$$

The line $(f^{\frac{1}{2}}u=y)$ meets the nodal cubic

$$u^2\left(\frac{u}{f^{\frac{1}{2}}}-hy\right)-2gu^2x+x^2(f^{\frac{1}{2}}u+y)=0$$

in the site of the node ($y = 0$, $u = 0$), and touches it in the above-named point of contact of (C) and (D).

For the second pair of points of contact of (C) and (D)

$$\left(\frac{x}{2h} = \frac{y}{g} = \frac{u}{2h^{\frac{1}{2}}}\right);$$

the corresponding complex is

$$(h^{\frac{1}{2}}u - x)\left[u^2\left(\frac{u}{h^{\frac{1}{2}}} - fx\right) - 2gu^2y + y^2(h^{\frac{1}{2}}u + x)\right] = 0.$$

The line meets and touches this nodal cubic, similarly to the former. There are four such complexes found by giving both signs to $f^{\frac{1}{2}}$, $h^{\frac{1}{2}}$.

Cor.—If f, h are both or singly negative, the linear factors are imaginary, and no complexes occur.

7. If the subsidiary curves (C) and (D) have double contact of the third order, the radii vectores through their points of contact coincide.

When this occurs, $g^2 = \frac{1}{3} = 4fh$.

With these values, (C) and (D) may be written, to show double contact at the pair of points ($f^{\frac{1}{2}}x = h^{\frac{1}{2}}y = f^{\frac{1}{2}}h^{\frac{1}{2}}u$),

$$3(xy - f^{\frac{1}{2}}h^{\frac{1}{2}}u^2)^2 - u^2(f^{\frac{1}{2}}x - h^{\frac{1}{2}}y)^2 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (C),$$

$$3(xy - f^{\frac{1}{2}}h^{\frac{1}{2}}u^2)^2 + u^2(f^{\frac{1}{2}}x - h^{\frac{1}{2}}y)^2 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (D).$$

The complex (A) has the parameters

$$\xi = c - 2gf^{\frac{1}{2}}, \quad \eta = d - 2gh^{\frac{1}{2}},$$

and consists of two lines, which were tangents at the nodes, and a conic, in which those lines intersect each other.

The quartic may take the two forms

$$u^4 - 2gu^3(f^{\frac{1}{2}}x + h^{\frac{1}{2}}y) + u^2(fx^2 + 2gxy + hy^2) - x^2y^2 = 0 \quad \ldots\ldots (A),$$

$$(x - h^{\frac{1}{2}}u)(y - f^{\frac{1}{2}}u)\left[(x - h^{\frac{1}{2}}u)(y - f^{\frac{1}{2}}u) + 2f^{\frac{1}{2}}u(x - h^{\frac{1}{2}}u) + 2h^{\frac{1}{2}}u(x - f^{\frac{1}{2}}u)\right] = 0$$
$$\ldots\ldots\ldots\ldots\ldots\ldots (A).$$

Two complexes are found, by giving both signs to $f^{\frac{1}{2}}$, $h^{\frac{1}{2}}$.

Cor.—Since the coordinates of every other point in (C), except the points of contact, render (D) positive, all other points in (C), (by § 4) are crunodes in the (A) system.

8. Two radii-vectores may unite, without necessary contact of the subsidiary curves (C) and (D).

In the equations of § 6, formed by addition and subtraction,

$$u^4(1+g^3-fh)-6gxyu^2+6x^2y^2=0,$$

$$u^2(1-g^3+fh)-2fx^2+2gxy-2hy^2=0,$$

write $u^2=rxy$, so that

$$r^2(1+g^3-fh)-6rg+6=0.$$

The vectors will coincide, if

$$r^2(1+fh-g^3)^2+4rg(1+fh-g^3)+4(g^3-4fh)=0.$$

The elimination of r determines the condition of coincidence

$$\{(g^3+5fh)(g^3-fh)-8g^3+14fh+3\}^2$$
$$=12(3fh-1)g^3(g^3-fh-1)(g^3-fh-5).$$

There are therefore eight cases to be examined for given values of f, h.

In general, cuspidal trinodal quartics (A) correspond; but if (C) and (D) touch each other quadruply, or doubly as in § 7, the quartics (A) are resolved into complexes.

9. This quartic in (g^3), when expanded, bears the form

$$g^8-4(7fh+1)g^6+2(39f^2h^2+78fh-1)g^4-4(19f^3h^3+3f^2h^2+77fh-3)g^2$$
$$+(fh-3)^2(5fh+1)^2=0$$

The invariants of this quartic are

$$I_4=\tfrac{64}{3}(81f^3h^3-9fh+1),$$

$$I_6=-256(81f^4h^4-45f^3h^3+9f^2h^2-fh+\tfrac{2}{7}).$$

The discriminant $(I_4^3-27I_6^2)$, after rejecting the factor 8^6, is

$$(3fh-1)^5(4fh-1).$$

10. When $3fh=1$, $(g^3-\frac{4}{3})^2(g^3-\frac{16}{3})^2=0$; two sets of values of g^3, and therefore of radii vectores, coincide.

Case (1), when $g^3=\frac{4}{3}$, is considered in § 7.

Case (2), when $g^3=\frac{16}{3}$: (C) and (D) may be written to show

double contact with each other, and with the conic $(u^3 = \frac{1}{3}\sqrt{3}\,xy)$; (C) and (D) also intersect in four other points,

$$(u^3-\tfrac{1}{3}\sqrt{3}\,xy)\ (u^3-3\sqrt{3}\,xy)-u^3\,(x\sqrt{f}-y\sqrt{h})^3 = 0 \ldots\ldots (C),$$

$$(u^3-\tfrac{1}{3}\sqrt{3}\,xy)(5u^3-3\sqrt{3}\,xy)+u^3\,(x\sqrt{f}-y\sqrt{h})^3 = 0 \ldots\ldots (D).$$

The corresponding quartics are resolved into two conics

$$u^3\left[\pm 2u+x\sqrt{f}+y\sqrt{h}\right]^2-(xy-u^3\sqrt{3})^2 = 0 \ldots\ldots\ldots\ldots (A).$$

11. When $4fh = 1$, the quartic of § 9 becomes

$$(g^3-\tfrac{9}{4})^2\left[g^3-(\tfrac{1}{2}\pm\sqrt{3})^2\right] = 0.$$

Case (1), when $g^3 = \frac{9}{4}$: two values of g^3 coincide, as in § 10, and double contact takes place, from the forms of (C) and (D); the (A) quartics are resolved into conics,

$$(u^3\mp xy)(u^3\mp 3xy)-u^3\,(x\sqrt{f}\mp y\sqrt{h})^3 = 0 \ldots\ldots\ldots\ldots (C),$$

$$(u^3\mp xy)(2u^3\mp 3xy)+u^3\,(x\sqrt{f}\mp y\sqrt{h})^3 = 0 \ldots\ldots\ldots\ldots (D),$$

$$\left[u\,(x\sqrt{f}\pm y\sqrt{h})-u^3\sqrt{2}\right]^2-(xy\mp u^2)^2 = 0 \ldots\ldots\ldots\ldots (A).$$

Case (2), when $g = \sqrt{3}-\frac{1}{2}$: (C) and (D) may take the forms

$$(u^3-\sqrt{3}\,xy)^2-u^3\,(x\sqrt{f}-y\sqrt{h})^3 = 0 \ldots\ldots\ldots\ldots (C),$$

$$(u^3-\sqrt{3}\,xy)\left[(3-\sqrt{3})\,u^3-\sqrt{3}\,xy\right]+u^3\,(x\sqrt{f}-y\sqrt{h})^3 = 0 \ldots (D).$$

Two vectors coincide without contact, when (C) is resolved into two conics, and (D) passes through four double points, which are the mutual intersections of those conics (two of them being the given nodes), as well as through four other points in (C). There are in all six, not eight, intersections (other than the given nodes), which are cusps on the corresponding (A) trinodals. Thus, the point $\left[x\sqrt{f} = y\sqrt{h} = u\,(12)^{-\frac{1}{4}}\right]$ is a cusp on

$$u^4-4\sqrt{2}\,u^3\,(f^{\frac{1}{2}}x+h^{\frac{1}{2}}y)\,3^{-\frac{1}{4}} = x^3y^3-u^3\left[(f^{\frac{1}{2}}x+h^{\frac{1}{2}}y)^2+xy(2\sqrt{3}-2)\right] \ldots (A),$$

or, as it may be written, to show the cusp,

$$\left[u-\frac{\sqrt{2}}{2}\,3^{\frac{1}{4}}\,(f^{\frac{1}{2}}x+h^{\frac{1}{2}}y)\right]^3\left[u-\frac{\sqrt{2}}{6}\,3^{\frac{1}{4}}\,(f^{\frac{1}{2}}x+h^{\frac{1}{2}}y)\right]$$

$$+(f^{\frac{1}{2}}x-h^{\frac{1}{2}}y)^2\left[u^3\,(1-\sqrt{3})+\tfrac{1}{4}\,(fx^2+6f^{\frac{1}{2}}h^{\frac{1}{2}}xy+hy^2)\right] = 0.$$

These examples will illustrate the three ways in which union of the vectors takes place.

I. *Bicrunodal Quartics.*

12. The constants $f = 1 = h$ will be used in the main and subsidiary equations, to facilitate computation, as also $u \equiv x+y-1$.

$$u^3(\xi x+\eta y-1) = x^2y^2-u^2(x^2+2gxy+y^2) \quad \text{.....(A)},$$

$$u^4+3x^2y^2-u^2(x^2+2gxy+y^2) = 0 \quad \text{.....(C)},$$

$$(g^2-1)u^4+3x^2y^2+u^2(x^2-4gxy+y^2) = 0 \quad \text{.....(D)}.$$

It is proposed to examine every case which occurs by varying (g), which is independent of the fixed nodes, by the methods already explained.

13. When $g = 1$: an ordinary case.

By eliminating u from (C) (Fig. 1), and (D) (Fig. 2), we obtain

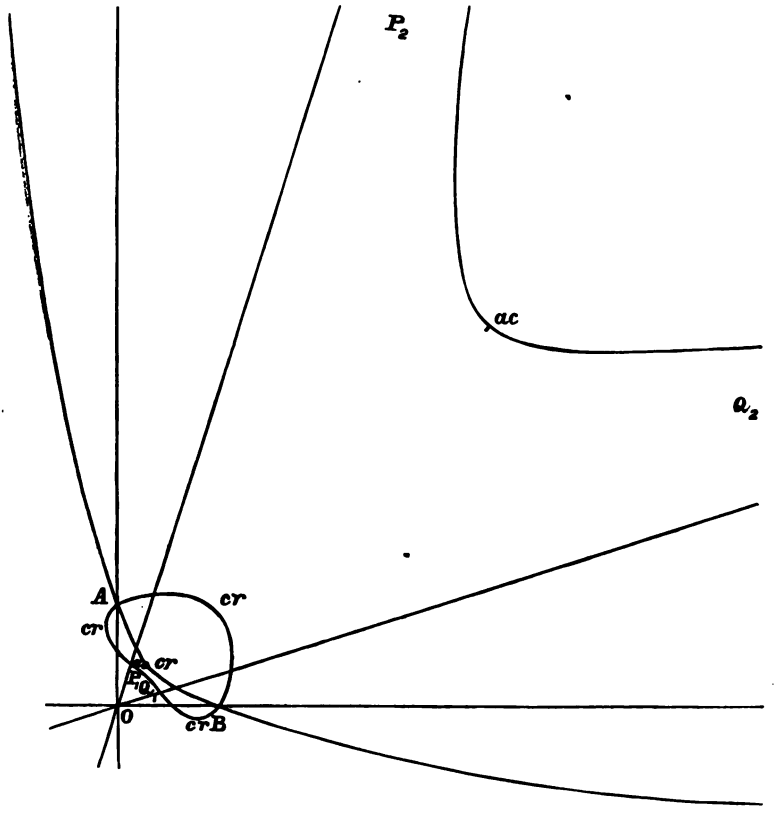


FIG. 1.

OP_1P_2, OQ_1Q_2, the vectors through their four real intersections

$$2x^2-(5+\sqrt{3})\,xy+2y^2=0,$$

or

$$(x-3{\cdot}037y)(y-3{\cdot}037x)=0.$$

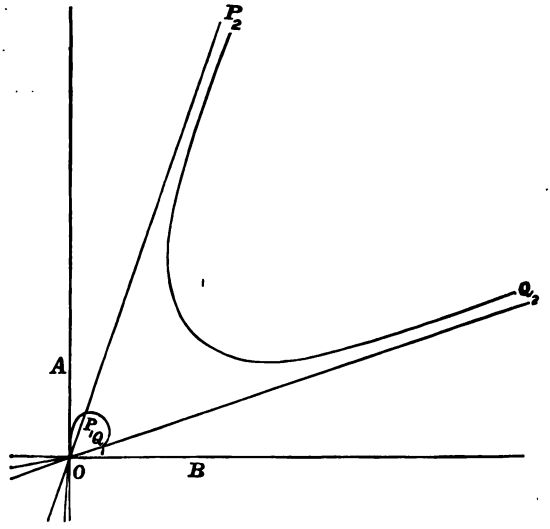


FIG. 2.

These four points P_1, Q_1; P_2, Q_2 are third (additional) cusps in the (A) system. All points on P_1 ac Q_1, P_2 ac Q_2 are third acnodes in (A); all other points on (C) are third crunodes in (A).

This is proved by the aid of (D), without determining, as in § 2, the values of ξ, η for a given point in (C), and drawing the curve (A). Make $x=y$ in (C):

$$(x-1)(3x-1)(x^2-4x+1)=0.$$

Of the apses thus found, the coordinates $(2\pm\sqrt{3},\ 2\pm\sqrt{3})$, when substituted in the function (D), render it negative,

$$3\,(2\pm\sqrt{3})^2-2\,(3\pm2\sqrt{3})^2 \quad \text{or} \quad -3\,(2\pm\sqrt{3})^2.$$

These apses are therefore acnodes. Similarly, the other apses are crunodes. And, since no alteration occurs in a locus of acnodes or of crunodes, except at the points of junction which are cusps, P_1 ac Q_1, P_2 ac Q_2 are loci of acnodes.

14. Quadruple contact occurs between (C) and (D), when $g=0$,

(§ 6), on the vectors $x = 0$, $y = 0$; the quartics (A) become complexes of lines and nodal cubics.

The four points of contact of

$$(u^2-x^2)(u^2-y^2)+2x^2y^2 = 0 \ldots\ldots\ldots\ldots\ldots\ldots \text{(C)},$$

$$(u^2-x^2)(u^2-y^2)-2x^2y^2 = 0 \ldots\ldots\ldots\ldots\ldots\ldots \text{(D)},$$

are $u \pm x = 0$, with $y = 0$; $u \pm y = 0$, with $x = 0$.

With the first pair (§ 2), $\xi = 1 \pm 2$, $\eta = 1$, (A) is resolved into the complex

$$(u \pm x)\left[u^2(u \pm x)+y^2(u \mp x)\right] = 0;$$

or, in full,
$$(x+y-1)^2(2x+y-1)+y^2(y-1) = 0,$$

$$(y-1)(x+y-1)^2+y^2(2x+y-1) = 0.$$

With the second pair of values, (A) is resolved into the complex

$$(u \pm y)\left[u^2(u \pm y)+x^2(u \mp y)\right].$$

The detached lines were tangents at nodes, and touch the associated nodal cubics in the four assigned points.

15. Double contact of the third order cannot exist between (C) and (D) when $f = h = 1$, but takes place when $f = h = \frac{1}{3} = \frac{g}{4}$. See § 7.

16. Next consider the values of g which cause two radii vectores to unite when (C) and (D) intersect each other. (§ 8.)

When $f = h = 1$, g has eight values in the equation

$$(g^4-4g^3+12)^2 = 24g^2(g^4-8g^3+12),$$

or
$$(g^2-6g+6)(g^2+6g+6)(g^2-2g-2)(g^2+2g-2) = 0.$$

When $\pm g = 3 \pm \sqrt{3}$, the curves (C) and (D) touch each other, and the corresponding (A) quartics are resolved into conics; when $\pm g = \sqrt{3} \pm 1$, (C) is resolved into conics, and (D) intersects it in six points, which are cusps in the (A) system. See § 11.

17. When $g = 3+\sqrt{3}$, and $(g-1)(5-g) = 1$. In this case

$$\left[u^2(g-1)-xy\right]\left[u^2(5-g)-3xy\right]-u^2(x-y)^2 = 0 \ldots\ldots\ldots \text{(C)},$$

$$\left[u^2(g-1)-xy\right]\left[u^2(g+1)-3xy\right]+u^2(x-y)^2 = 0 \ldots\ldots\ldots \text{(D)}.$$

They are seen to have double contact with each other, and with the intermediate ellipse $\left[u^2(g-1)-xy\right]$, at the points

$$\left[x = y = \pm u\sqrt{(g-1)}\right].$$

The bicrunodal (C) consists of two circuits, which intersect in the common nodes; the biacnodal (D) of two elongated circuits, one of which is unifolium, and each touches a separate circuit of (C).

The parameters (§ 2) which correspond to these points of contact are $\xi = \eta = 1 \mp 4\sqrt{(g-1)}$.

The corresponding (A) quartics are resolved into conics:

$$u^2\left[u \mp 4(x+y)\sqrt{(g-1)}\right] = x^2y^2 - u^2(x^2+2gxy+y^2),$$

or
$$\left[xy - u^2(g-1)\right]^2 = u^2\left[x+y \mp 2u\sqrt{(g-1)}\right]^2.$$

The first consists of two ellipses, the second of an ellipse and hyperbola. All pass through the common nodes, and each pair touches each other in the points $\left[x = y = \pm u\sqrt{(g-1)}\right]$.

The curves (C) and (D) intersect each other in four real points, in which the conic $(u^2 = xy)$ is met by the vectors

$$\left[x^2+y^2+(2g-4)xy = 0\right].$$

There are therefore acnodes, cusps, and crunodes in (A).

18. When $g = 3-\sqrt{3}$, and $(g-1)(5-g) = 1$. The results of § 17 are equally true, with a modified interpretation.

The curve (C) consists of two intersecting circuits, each of which touches a separate circuit of (D); these last are short and long, without asymptotes. The two pairs of conic complexes (A) are an ellipse and hyperbola, which pass through the nodes and touch each other in pairs, as in § 17. But the vectors $\left[x^2+y^2+(2g-4)xy\right]$ are imaginary, so that (C) and (D) do not intersect; all points in (C) are therefore crunodes in (A), except the points $\left[x = y = \pm u\sqrt{(g-1)}\right]$.

19. When $-g = 3+\sqrt{3}$, the forms of the equations to the subsidiary curves may be derived from § 17, by changing the sign of xy, and leaving (g) unaltered:

$$\left[u^2(g-1)+xy\right]\left[u^2(5-g)+3xy\right] - u^2(x+y)^2 = 0 \ldots\ldots\ldots (C),$$

$$\left[u^2(g-1)+xy\right]\left[u^2(g+1)+3xy\right] + u^2(x+y)^2 = 0 \ldots\ldots\ldots (D).$$

They have double contact at the points $\left[x = -y = \pm u\sqrt{(g-1)}\right]$.

The corresponding quartics (A) may be resolved into pairs of conics

$$\left[u^2(g-1)+xy\right]^2 = u^2\left[2u\sqrt{(g-1)} \mp (x-y)\right]^2.$$

Cor.—When $-g = 3-\sqrt{3}$, the results of § 19 are applicable, with the altered value of (g).

20. When $g = \sqrt{3}+1$, the subsidiary curves only meet in the given nodes; all points in (C) are (third) crunodes in the (A) system. In this case

$$(u^2 - \sqrt{3}xy)^2 - u^2(x+y)^2 = 0 \ldots\ldots\ldots\ldots\ldots\ldots (C),$$

$$\sqrt{3}(u^2 - \sqrt{3}xy)\left[(2+\sqrt{3})u^2 - xy\right] + u^2(x+y)^2 = 0 \ldots\ldots (D).$$

The contact is imaginary, as well as the intersections.

Cor.—These results are applicable when $-g = \sqrt{3}+1$, by changing the sign of xy, and leaving g unaltered.

21. When $g = \sqrt{3}-1$, (C) and (D) meet in six points, besides the given nodes:

$$(u^2 - \sqrt{3}xy)^2 - u^2(x-y)^2 = 0 \ldots\ldots\ldots\ldots\ldots (C),$$

$$\sqrt{3}(u^2 - \sqrt{3}xy)\left[(2-\sqrt{3})u^2 + xy\right] - u^2(x-y)^2 = 0 \ldots\ldots (D).$$

(D) consists of a single circuit, and each hyperbolic branch meets (C) in three points, of which one is double, without contact at any, as in § 11 (2). There are six points of intersection on (C), which are cusps on (A); and internal points on (C) between two cusps are acnodes on (A); the rest are crunodes. Thus, for the pair of points $\left[x = y = \pm 3^{-\frac{1}{2}}u\right]$,

$$\xi = \eta = 1 \mp 4(3^{-\frac{1}{2}})$$

in (B). The corresponding quartics (A), have (third) cusps

$$u^4 \mp 4(x+y)u^3 3^{-\frac{1}{2}} = x^2y^2 - u^2\left[(x+y)^2 + 2xy(\sqrt{3}-2)\right];$$

or, as it may be written,

$$\left[u \mp \tfrac{1}{2}3^{\frac{1}{2}}(x+y)\right]^3\left[u \pm \tfrac{1}{6}3^{\frac{1}{2}}(x+y)\right] + (x-y)^2\left[u^2(1-\tfrac{1}{2}\sqrt{3}) + \tfrac{1}{16}(x^2+6xy+y^2)\right] = 0.$$

These trinodal quartics have cusps on one of the two intersecting circuits, of which they are composed, situated on the coincident radii vectores.

22. Let $-g = \sqrt{3}-1$. *Mutatis mutandis*, as in § 19,

$$(u^2 + \sqrt{3}xy)^2 = u^2(x+y)^2 \ldots\ldots\ldots\ldots\ldots (C),$$

$$\sqrt{3}(u^2 + \sqrt{3}xy)\left[u^2(2-\sqrt{3}) - xy\right] = u^2(x+y)^2 \ldots\ldots (D).$$

The curve (D) has a single bifolium circuit, which intersects (C) in four single points, and two double points, besides the acnodes. There

are, therefore, six cusps in the (A) binodal system; of which two are $\left[u \mp \frac{1}{2}3^{\frac{1}{2}}(x-y)\right]^3 \left[u \pm \frac{1}{6}3^{\frac{1}{2}}(x+y)\right]$

$$+(x+y)^2\left[u^2(1-\tfrac{1}{2}\sqrt{3})+\tfrac{1}{16}(x^2+6xy+y^2)\right]=0.$$

23. More generally, the number of cusps can be determined for an assigned value of (g).

The equations (C) and (D) may be combined into the forms

$$gu^2 = \sqrt{3}(\sqrt{3}\mp 1)xy,$$

$$2g(x^2+y^2)+(1\mp\sqrt{3})(g^2\pm 2\sqrt{3})xy=0.$$

It is thus found that, if numerically $\pm g > 3+\sqrt{3}$, or $< \sqrt{3}-1$, (C) and (D) intersect in eight points; if $\pm g < 3+\sqrt{3}$ and $> \sqrt{3}-1$, they meet in four, unless $\pm g = \sqrt{3}+1$, when they meet in no point but the nodes. In these several cases, there are eight, four, and no (third) cusp in the (A) binodal system.

II. *Biacnodal Quartics.*

24. In this system f, h are negative, and assumed to be unity in the main and subsidiary equations; $u \equiv x+y-1$.

$$u^3(\xi x+\eta y-1) = x^2y^2+u^2(x^2+2gxy+y^2) \quad \text{.....(A),}$$

$$u^4+3x^2y^2+u^2(x^2+2gxy+y^2)=0 \quad \text{.....(C),}$$

$$(g^2-1)u^4+3x^2y^2-u^2(x^2-4gxy+y^2)=0 \quad \text{.....(D).}$$

It is seen that any value of $\pm g$, numerically $< \sqrt{3}+1$, reduces (C) to the acnodes (1, 0), (0, 1), when it is written

$$(u^2+\sqrt{3}xy)^2+u^2\left[(x+y)^2+2(g-\sqrt{3}-1)xy\right]=0,$$

Consequently there are no trinodal quartics in the (A) system.

25. When $g=\sqrt{3}+1$, (C) consists of the given acnodes and two others $\left[x=-y=\pm 3^{-\frac{1}{2}}u\right]$. For this last pair the corresponding (A) quartics

$$\left[u\mp\tfrac{1}{2}3^{\frac{1}{2}}(x-y)\right]^3\left[u\pm\tfrac{1}{6}3^{\frac{1}{2}}(x-y)\right]$$

$$+(x+y)^2\left[-u^2(1+\tfrac{1}{2}\sqrt{3})+\tfrac{1}{16}(x^2+6xy+y^2)\right]=0$$

exhibit (third) cusps in the second and fourth quadrants.

The values of the parameters ξ, η in these (A) quartics are, by § 3,

$$\xi = 1\mp 4(3^{-\frac{1}{2}}), \quad \eta = 1\pm 4(3^{-\frac{1}{2}}).$$

26. When $-g = \sqrt{3}+1$, the cusped trinodal (A) quartics are

$$\left[u \mp \tfrac{1}{2} 3^{\frac{1}{2}} (x+y)\right]^2 \left[u \pm \tfrac{1}{6} 3^{\frac{1}{2}} (x+y)\right] + (x-y)^2 \left[-u^2 (1+\tfrac{1}{2}\sqrt{3}) + \tfrac{1}{12} (x^2+6xy+y^2)\right] = 0.$$

These four are the only trinodal quartics when $\pm g = \sqrt{3}+1$.

27. The biacnodal quartics (A) can never degenerate into complexes, since the subsidiary curves (C) and (D) cannot touch each other.

In §§ 6, 7 it was shown that (C) and (D) cannot have quadruple contact, or double contact of the third order. They can only have double contact when $\pm g = \sqrt{3}+3$, and it will be proved that this contact is imaginary.

28. Let $g = 3+\sqrt{3}$, $(g-1)(5-g) = 1$. As in § 17, (C) and (D) may be written, to exhibit imaginary contact,

$$\left[u^2 (g-1)+xy\right]\left[u^2 (5-g)+3xy\right]+u^2 (x-y)^2 = 0 \quad \text{.....(C)},$$

$$\left[u^2 (g-1)+xy\right]\left[u^2 (g+1)+3xy\right]-u^2 (x-y)^2 = 0 \quad \text{.....(D)}.$$

They intersect in four real points, in which the vectors

$$x^2+(2g-4)\,xy+y^2 = 0$$

intersect the conic $u^2+xy = 0$.

When $-g = 3+\sqrt{3}$, like results are obtained by writing $-xy$ for xy, and leaving (g) unaltered.

29. For all values of $(\pm g)$ numerically $> \sqrt{3}+1$, (C) and (D) intersect in four real points, which are cusps in the corresponding (A) quartics. (C) is therefore a locus of crunodes, cusps, and acnodes in (A).

Let $g = 3$. (Fig. 3.) The locus of nodes (C) consists of two unifolium circuits besides the acnodes A, B; and each circuit is divided into two portions by branches of the bicrunodal quartic (D). P ac Q is the locus of acnodes, P cr Q of crunodes, while P, Q are cusps in the corresponding (A) quartics.

Ex.—The point $(1, -1)$ lies in P ac Q; the corresponding parameters (§ 3) are $3, -1$; the corresponding (A) quartic has $(1, -1)$ for a third acnode

$$u^2 (3x-y+1) = x^2y^2+u^2 (x^2+6xy+y^2) \quad \text{.....(A)}.$$

Ex.—The point $(\frac{1}{3}\sqrt{3}, -\frac{1}{3}\sqrt{3})$ lies on P or Q, and is a crunode on

$$u^4+10u^3(x-y)3^{-\frac{1}{2}} = x^2y^2+u^2(x^2+6xy+y^2) \quad \text{.....(A).}$$

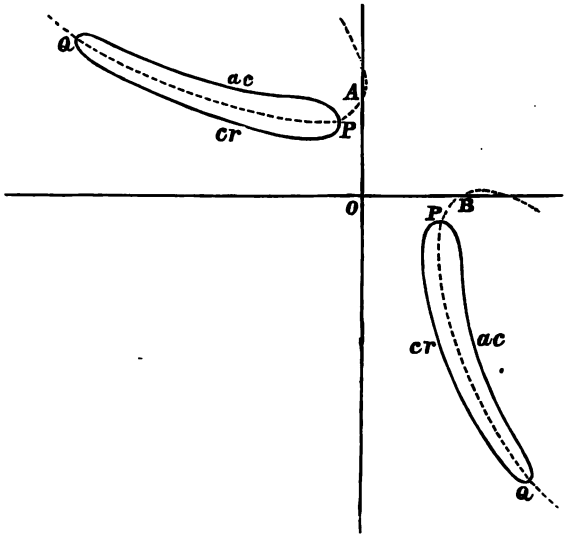


FIG. 3.

III. *Quartics with one crunode and one acnode.*

30. In this system f is positive, h negative, and assumed to be unity.

$$u^3(\xi x+\eta y-1) = x^2y^2+u^2(x^2+2gxy-y^2) \quad \text{.....(A),}$$

$$u^4+3x^2y^2+u^2(x^2+2gxy-y^2) = 0 \quad \text{.....(C),}$$

$$(g^2+1)u^4+3x^2y^2-u^2(x^2-4gxy-y^2) = 0 \quad \text{.....(D).}$$

$$u \equiv x+y-1,$$

as before.

The subsidiary curves (C) and (D) cannot have double contact of the third order (§ 7), but they have quadruple contact (§ 6), when $g^2 = 4$. The points of contact are thus defined:

$$2u^2+gxy = 0, \quad x^2-y^2 = 0.$$

31. Let $g = 2$. The bisector $(x+y = 0)$ passes through P, Q, $(-1, 1)$, $(1, -1)$, in which (C) and (D) touch each other; the other

points of contact are imaginary. The curve (C) consists of a crunodal circuit with an acnode at B. (Fig. 4.) The curve (D) has its crunode at B, and acnode at A.

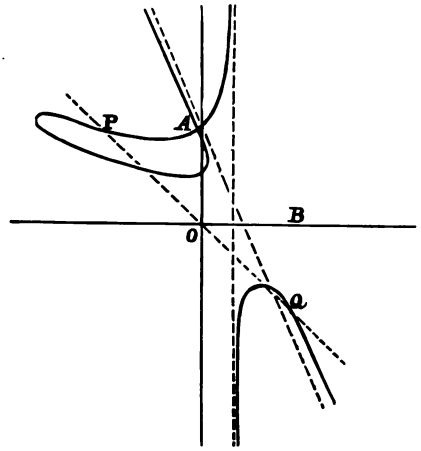


FIG. 4.

To the point $P(-1, 1)$ the parameters are $\xi = 1$, $\eta = 5$; and the (A) quartic

$$u^3(x+5y-1) = x^2y^2+u^2(x^2+4xy-y^2)$$

is resolved into an acnodal cubic, and a line $(y = 1)$ which touches it at P.

$$(u^2+y^2)(u^2-x^2)+4u^2y(u-x) = 0.$$

For $Q(1, -1)$, the corresponding complex is

$$(u^2+y^2)(u^2-x^2)-4u^2y(u+x) = 0.$$

32. Let $g = -2$. The internal bisector $(x = y)$ passes through the points of real contact of (C) and (D), $(1, 1)$, $(\frac{1}{3}, \frac{1}{3})$.

The corresponding (A) complexes are

$$(u-x)\left[(u+x)(u^2+y^2)-4u^2y\right] = 0,$$

$$(u+x)\left[(u-x)(u^2+y^2)+4u^2y\right] = 0.$$

Each acnodal cubic is touched by a linear factor, which represents a tangent at a node, in one of the specified points.

33. Every point on (C), besides P and Q, is a second crunode in th[illegible]

(A) system. For, if the coordinates of two assigned points on (C), ($x^2 = y^2 = \frac{1}{3}$), be substituted in (D), that function is positive (§ 4).

34. If $g^2 < 4$, the curves (C) and (D) do not meet. Every point on (C) is therefore a second crunode on (A), for the reasoning of § 33 is applicable to the modified forms.

35. If $g^2 > 4$, (C) and (D) intersect in four real points. Let $g = 3$. Through the real intersections of the curves

$$u^4 + 3x^2y^2 + u^2(x^2 + 6xy - y^2) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (C),$$

$$10u^4 + 3x^2y^2 - u^2(x^2 - 12xy - y^2) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (D),$$

the two vectors $(x + 2{\cdot}645y)(x + {\cdot}6465y) = 0$ pass.

These vectors intercept in (C) (see Fig. 4, modified) on AP, that part of the loop which is remote from O, and on Q, two series of points which are acnodes in (A); all other points are crunodes, except the four points of intersection, which are cusps in (A).

36. In this system, two radii vectores cannot unite (§ 8), unless (C) and (D) touch each other.

For the values of g^2 necessary for such union, when $f = -h = 1$, are (by § 9) given by the quartic

$$g^8 + 24g^6 - 80g^4 + 384g^2 + 256 = 0;$$

but this has no positive value.

Moreover, no lower value of $(-fh)$ will give positive values of g^2 in the corresponding quartic.

This result is confirmed by the discriminant of the quartic in g^2 (§ 9), which only yields two positive limiting values of fh.

IV. *Quartics with a crunode and a cusp.*

37. In this system one of the constants is zero, and the other negative, and here assumed to be unity. The quartics (A), (C), (D) are

$$u^3(\xi x + \eta y - 1) = x^3y^2 - u^2(x^2 + 2gxy) \quad \ldots\ldots\ldots\ldots\ldots\ldots (A),$$

$$u^4 + 3x^2y^2 - u^2(x^2 + 2gxy) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (C),$$

$$g^2u^4 + 3x^2y^2 + u^2(x^2 - 4gxy) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (D),$$

$$u \equiv x + y - 1.$$

The curves (C) and (D) touch at their common tacnode (0, 1), which is always real in (D), but imaginary in (C), if $g^2 < 3$.

By combining (C) and (D) in the forms

$$6x^2y^2 - 6gxyu^2 + (g^2+1)\,u^4 = 0,$$

$$(g^2-1)\,u^2 + 2x^2 - 2gxy = 0,$$

we see that (C) and (D) intersect in four points, if $g^2 > 2$, and touch in four points at the tacnode, if $g^2 = 2$ (see § 6). If $g^2 = 3$, two of the four points are merged in the tacnode (Fig. 7).

38. Let $g^2 = 2$; then $u^2 = gxy$, and $gx^2 = xy$. (Figs. 5 and 6.)

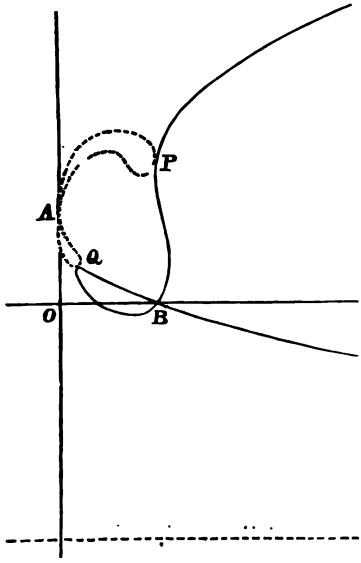


Fig. 5.

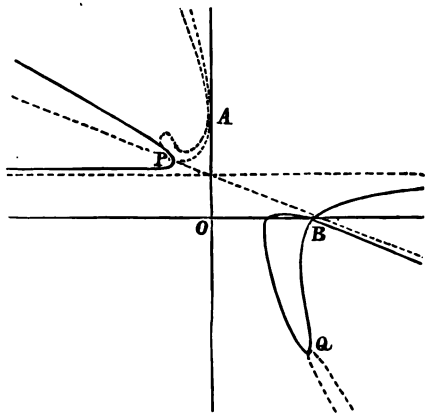


Fig. 6.

The curves (C) and (D) have double contact at the points P, Q ($gx = y = \pm u$).

The four corresponding (A) quartics

$$u^2\left[(1-2g)\,x - 1\right] = x^2y^2 - u^2\,(x^2+2gxy),$$

$$u^2\left[(1+2g)\,x + 2y - 1\right] = x^2y^2 - u^2\,(x^2+2gxy),$$

are resolved into complexes of lines and cubics

$$(x-1)\left[u^3-2gxu^2+x^2(u+y)\right]=0,$$
$$(u+y)\left[u^3+2gxu^2+x^2(x-1)\right]=0.$$

The cubics are cusped at A, and touched by the lines in P, Q.

39. When $g^2=3$, (C) and (D) intersect in two points only, besides the common nodes A, B. (Fig. 7.) Two radii vectores do not unite,

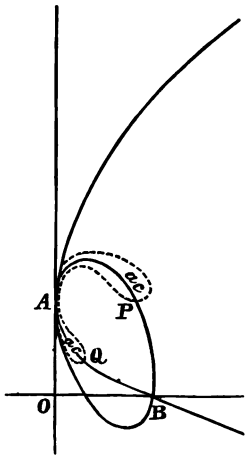


FIG. 7.

as it might be inferred from the formula of § 9,

$$(g^2+1)^2(g^2-3)^2=0.$$

That formula is inapplicable, since it was obtained on the hypothesis that (h) was constant.

In this case $(u^2-gxy)^2-u^2x^2=0$ .....(C),

$$(u^2-gxy)(3u^2-gxy)+u^2x^2=0 \text{ .....(D).}$$

By combining, $(u^2-gxy)(2u^2-gxy)=0,$

$$u^2(u^2-gxy+x^2)=0,$$

and the vectors are $x^2=0$, $x^2-2gxy=0$. [? $2x^2-gxy=0$.]

The conics have a contact of five points with (D) at the tacnode, and meet (D) in the points $(gy=2x=\pm 2u)$.

These points are cusps in the corresponding (A) trinodal quartics

$$u^4\mp\tfrac{1}{3}u^3(10x+2gy)=x^2y^2-u^2(x^2+2gxy) \text{ .....(A).}$$

These may be written to show both cusps as well as the crunode, and the bitangent, which touches a folium at points situated on the axes

$$(u-x)^2(3u-x-2gy) = x^2(3u-x-gy)^2,$$
$$(u+x)^2(3u+x+2gy) = x^2(3u+x+gy)^2.$$

There are four bicuspidal and singly crunodal quartics, since (g) has two signs. Only one figure of (C) and (D) is drawn in illustration. The portion of the ellipse, intercepted by the upper lobe of (D), and the portion of the hyperbola, intercepted by the lower lobe of (D), yield acnodes on (A).

40. For all values of $g^2>2$, (C) is a locus of acnodes, cusps, and crunodes on the corresponding (A) trinodal quartics.

V. *Quartics with an acnode and a cusp.*

41. In this system f is positive, and here taken to be unity;

$$u \equiv x+y-1,$$
$$u^2(\xi x+\eta y-1) = x^2y^2+u^2(x^2+2gxy) \quad \text{.....(A)},$$
$$u^4+3x^2y^2+u^2(x^2+2gxy) = 0 \quad \text{.....(C)},$$
$$g^2u^4+3x^2y^2-u^2(x^2-4gxy) = 0 \quad \text{.....(D)}.$$

If $g^2 =$ or <3, (C) is reduced to the node and tacnode. There are consequently no critical values of the parameters ξ, η (§ 3), and no trinodal quartics in the system (A). The curves (C) and (D) touch at their common real tacnode, when $g^2 =$ or >3.

42. If $g^2>3$, (C) and (D) intersect each other in four points, two of which are imaginary. All points on (C) which lie between the (y) axis and these points of intersection, are acnodes on the corresponding (A) quartics, and are thus separated by cusps from the other points, which are crunodes.

Ex. $g=3$. The curves (C), (D) meet in two of the points in which the conics

$$(u^2+\cdot4417xy)(u^2+1\cdot358xy) = 0$$

are intersected by the conics

$$4u^2-x^2+3xy = 0,$$

or by the vectors

$$x(x-1\cdot233y) = 0, \quad x(x+2\cdot433y) = 0.$$

The last pair only gives real intersections, which discriminate the nodes on (C). Both (C) and (D) have real tacnodes; the acnodal curve (C) has two asymptotes, the crunodal (D) has four.

VI. *Bicuspidal Quartics.*

43. The general system is

$$u^3(\xi x+\eta y-1) = x^3y^3+2gxyu^2 \quad \text{.....(A).}$$

The locus of nodes $u^4+2u^2gxy+3x^2y^2 = 0$ .....(C).

The subsidiary curve $(gu^2+xy)(gu^2+3xy) = 0$.....(D).

The parameters ξ, η, and the discriminating curve (B), which is their envelop, are thus determined.

By § 3, $\frac{1}{y}(\xi-c) = \frac{1}{x}(\eta-d) = \frac{2}{u}\left(\frac{xy}{u^2}+g\right) = \frac{2}{u}(G+g)$,

since, by (C), $xy = \frac{1}{3}u^2\left[-g\pm\sqrt{(g^2-3)}\right] = Gu^2$, suppose.

Eliminate x, y from these equations.

$$(\xi-c)(\eta-d) = 4G(G+g)^2 \quad \text{.....(B).}$$

44. The line $(\xi x+\eta y = 1)$ intersects each curve (A) in the coordinate axes, and may be regarded as a satellite of the line of nodes $(u \equiv cx+dy-1)$, of which (B) is the envelop. All conjoint values of ξ, η, which satisfy (B), determine a third node in the quartics (A); all slightly varying values of ξ, η, which denote lines external to (B), or intersecting it, denote quartics in (A), which are companion-curves to the preceding, but not trinodal. See below, § 47, with Figs. 9, 10.

When $g^2<3$, the locus (C) is resolved into two imaginary conics; there are therefore no trinodal values in the (A) system.

45. When $g^2 = 3$, (C) becomes two coincident conics

$$(u^2+gxy)^2 = 0;$$

(D) two distinct conics

$$(u^2+gxy)(gu^2+xy) = 0.$$

Every point in (C) is also on (D), and is therefore a cusp in a corresponding curve in (A). For a point in the locus $(U^2+gXY=0)$, the parameters ξ, η are (if $c = d = 1$)

$$\xi-1 : \eta-1 : 4 :: Y : X : gU.$$

The corresponding quartic (A) is tricuspidal,

$$u^4+\frac{4}{gU}u^3(xY+yX)-2gu^2xy-x^2y^2=0,$$

and may be written, to exhibit the third cusp (X, Y, U),

$$\left[uU+\frac{g}{2}(xY+yX)\right]^3\left[uU-\frac{g}{6}(xY+yX)\right]$$
$$+\tfrac{3}{16}(xY-yX)^2(x^2Y^2+6xyXY+y^2X^2-8u^2U^2)=0.$$

Ex.—Let $g=-\sqrt{3}$, and let the points in (C) be the apses

$$\pm u=x\sqrt[4]{3}=y\sqrt[4]{3}=\sqrt[4]{3}\,(2\mp\sqrt[4]{3})^{-1}.$$

The corresponding coordinates in the discriminating curve (B) are

$$\xi-1=\eta-1=\mp1{\cdot}755.$$

With these parameters, the corresponding tricuspidal quartics (A) are

$$u^4\mp1{\cdot}755u^3(x+y)=x^2y^2-2\sqrt{3}\,u^2xy.$$

For convenient calculation these may be written

$$(xy-\sqrt{3}\,u^2)^2=2{\cdot}245u^4-1{\cdot}755u^3 \text{ or } 5{\cdot}755u^4+1{\cdot}755u^3.$$

The former is drawn in Fig. 8, with two companion-curves, which are obtained by slightly differing parameters ξ, η.

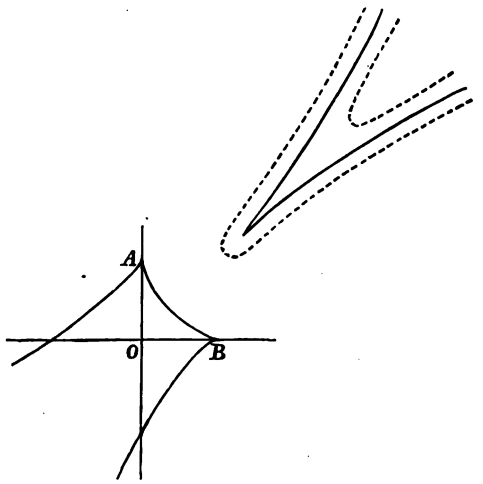


FIG. 8.

46. When $\xi = \eta = 1$, the quartic (A) is resolved into two conics.

In this case $(xy+gu^2=0)$ is a necessary condition, and must be a factor of (C). This is only possible when $g^2+1=0$. See § 8, where the condition $\left[(g^2+1)(g^2-3)=0\right]^2$ is shown to be necessary for the imaginary contact of (C) and (D), when $fh=0$.

The terminal bicuspidal quartic (A),

$$x^2y^2-4xyu^2+u^4=0,$$

has two asymptotes parallel to the axes.

47. For values of $g^2>3$, (C) and (D) are resolved into two real conics; all points on that portion of (C) which lies between the two portions of (D) are acnodes in (A); and all points in the external portion of (C) are crunodes in (A).

Let $g=-2$. The subsidiary curves (C), (D) are

$$(u^2-xy)(u^2-3xy)=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(C)},$$

$$(2u^2-xy)(2u^2-3xy)=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(D)}.$$

The (B) curves for the portions of (C) are severally

$$(\xi-1)(\eta-1)=4 \text{ and } \tfrac{100}{27}.$$

Ex. 1.—To the point $(x=y=u=1)$ the line $(\xi=\eta=-1)$ corresponds.

The point is an acnode on the corresponding (A) quartic

$$u^4+2u^3-4xyu^2+x^2y^2=0.$$

The companion-curves are unipartite, no trace of a separate circuit being found.

Ex. 2. — To the point $(x=y=-u=\frac{1}{3})$ the line $(\xi=\eta=3)$ [corresponds], and [the point is on] the acnodal (A) quartic

$$3u^4+2u^3+4u^2xy-x^2y^2=0.$$

Ex. 3. $\xi=\eta=1-\frac{10}{9}\sqrt{3}$. The point $(x=y=\frac{1}{3}\sqrt{3}\,u=2+\sqrt{3})$ is a crunode on the quartic (A), Fig. 9,

$$u^4-\tfrac{10}{9}\sqrt{3}\,(x+y)\,u^3=x^2y^2-4u^2xy,$$

or

$$3{\cdot}0755u^4-1{\cdot}9245u^3=(xy-2u^2)^2.$$

The companion-curves are drawn for slightly differing values of the

conjoint-parameters, *i.e.*, as the line (ξ, η) does or does not intersect the discriminating (B) curve. They are both unipartite.

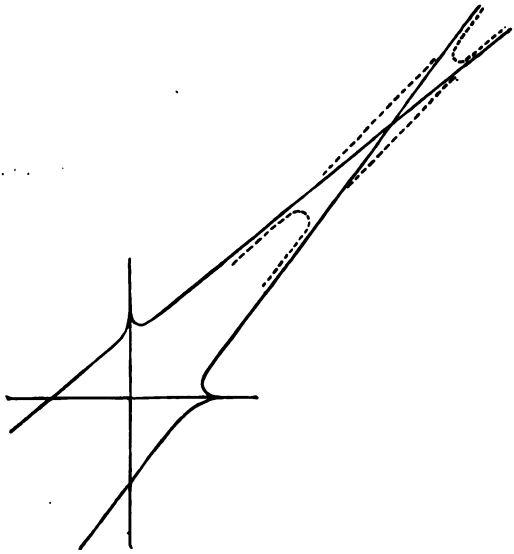


FIG. 9.

Ex. 4. $\xi = \eta = 1 + \frac{10}{9}\sqrt{3}$. The point $(x = y = 2 - \sqrt{3})$ is a crunode on

$$(xy - 2u^2)^2 = 1{\cdot}9245u^3 + 6{\cdot}9245u^4 \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(A)}.$$

(Fig. 10.) The unipartite and bipartite companion-curves are also drawn.

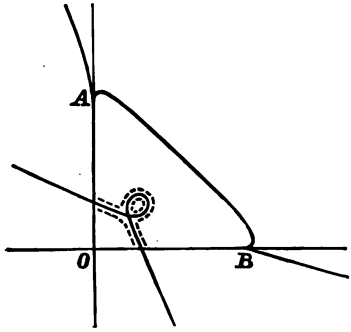


FIG. 10.

Bicuspidal Quartics, when both cusps are imaginary.

48. The general system of such quartics, and their subsidiary curves, may be thus denoted :

$$u^3(\xi x+\eta y-1) = (mx^2+ny^2)^2+2gu^3(mx^2+ny^2) \quad \ldots\ldots\ldots(A),$$

$$u^4+2u^3g(mx^2+ny^2)+3(mx^2+ny^2)^2 = 0 \quad \ldots\ldots\ldots(C),$$

$$(gu^2+mx^2+ny^2)(\tfrac{1}{3}gu^2+mx^2+ny^2) = 0 \ldots\ldots\ldots\ldots(D).$$

The discriminating curve is found, as in § 43, to consist of two conics

$$\frac{1}{m}(\xi-c)^2+\frac{1}{n}(\eta-d)^2 = 16G(G+g) \ldots\ldots\ldots\ldots(B),$$

where $$3G^2+2gG+1 = 0.$$

When $g^2<3$, the locus of nodes (C) is resolved into two imaginary conics, and the system (A) has no trinodal quartics.

49. When $g^2 = 3$, the subsidiary curves are

$$\left[u^2+g(mx^2+ny^2)\right]^2 = 0 \ldots\ldots\ldots\ldots\ldots\ldots(C),$$

$$\left[u^2+g(mx^2+ny^2)\right]\left[gu^2+mx^2+ny^2\right] = 0 \ldots\ldots\ldots(D).$$

Every point in (C) is also in (D), and therefore is a cusp in (A).

The parameters

$$\xi = c+\frac{1}{Ug}8mX, \qquad \eta = d+\frac{1}{Ug}8nY$$

correspond to a point (X, Y, U) on the conic

$$U^2+g(mX^2+nY^2) = 0,$$

and

$$\frac{1}{m}(\xi-c)^2+\frac{1}{n}(\eta-d)^2+\frac{64}{g^3} = 0 \ldots\ldots\ldots\ldots(B).$$

The corresponding tricuspidal quartic (A),

$$u^4+\frac{1}{Ug}8u^3(mxX+nyY)-2gu^2(mx^2+ny^2)-(mx^2+ny^2)^2 = 0,$$

may be written, to show the third cusp (X, Y, U),

$$\left[uU+g(mxX+nyY)\right]^3\left[uU-\frac{g}{3}(mxX+nyY)\right]$$

$$-g^3mn(xY-yX)^2\left[(mx^2+ny^2)(mX^2+nY^2)+(mxX+nyY)^2-2u^2U^2\right]$$

$$= 0.$$

Ex. $g = -\sqrt{3}$, $m = n = 1 = c = d$. The quartic

$$u^4 - 8\,(108)^{-\frac{1}{2}}\,u^3\,(x+y) = (x^2+y^2)^2 - 2\sqrt{3}\,u^2\,(x^2+y^2)$$

is tricuspidal, whose real cusp is $U : X : Y :: \sqrt[4]{(12)} : 1 : 1$.

50. For higher values of g^2, (C) and (D) are resolved into pairs of conics.

Ex. $g = -2$. Of the pairs of conics

$$(x^2+y^2-u^2)(x^2+y^2-\tfrac{1}{2}u^2) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (C),$$

the hyperbola is interior, the ellipse exterior, to the pair

$$(x^2+y^2-2u^2)(x^2+y^2-\tfrac{3}{2}u^2) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (D).$$

All points in the former conic are acnodes, in the latter crunodes, in (A).

51. If (u) is constant, the (A) system consists of Cartesians. The locus of nodes (C) takes the modified form

$$u^3 = 3\,(mx^2+ny^2)^2 + 2gu^2\,(mx^2+ny^2).$$

The subsidiary curve (D) is unaltered.

52. Binodal quartics, when both nodes are imaginary, are not adapted for this process of classification. In order to classify bicircular quartics with collinear foci, I selected different parameters (*Proceedings of Lond. Math. Soc.*, Vol. XIV.).

53. Binodal quartics, when both nodes are real, and one of them is at infinity, are classified, as in the preceding paragraphs, by writing $dy-1$ for (u), and equating (C) to zero; but the forms must be modified when both nodes are at infinity.

Binodal Quartics, with two nodes at infinity.

54. If the same form, as in § 1, [be taken] when (u) is constant, for such a system

$$u^3\,(\xi x+\eta y-1) = x^2y^2 - u^2\,(fx^2+2gxy+hy^2) \quad \ldots\ldots\ldots\ldots (A).$$

For critical values
$$\left.\begin{aligned} u^3\xi &= 2xy^2 - 2u^2\,(fx+gy) \\ u^3\eta &= 2x^2y - 2u^2\,(gx+hy) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (B),$$

$$u^3 = 3x^2y^2 - u^2\,(fx^2+2gxy+hy^2) \quad \ldots\ldots\ldots\ldots\ldots\ldots (C),$$

and, without alteration,

$$u^4\,(g^2-fh) + u^2\,(fx^2-4gxy+hy^2) + 3x^2y^2 = 0 \quad \ldots\ldots\ldots\ldots (D).$$

These last intersect in the hyperbola

$$(2xy-u^3g)^2 = \tfrac{1}{3}u^3\left[u\,(g^3+2fh)+2\right].$$

55. If these hyperbolæ coincide, (C) and (D) have quadruple contact, and the corresponding (A) quartic is resolved into a straight line and cubic.

In this case $2xy = u^3g, \quad u\,(g^3+2fh)+2 = 0,$

and the four points of contact

$$x : y : u :: g : 2f : 2f^{\frac{1}{2}}, \quad x : y : u :: 2h : g : 2h^{\frac{1}{2}}.$$

For the first pair of points the corresponding complexes are

$$2u^3xf^{\frac{1}{2}}g+u^3y\,(ufh-1)f^{-\frac{1}{2}}+u^3-u^2\,(fx^2+2gxy+hy^2)+x^2y^2 = 0 \ldots (A),$$

or $\quad (y-uf^{\frac{1}{2}})\left[x^2y+ux^2f^{\frac{1}{2}}-u^2\,(2gx+hy)-u^2f^{-\frac{1}{2}}\right] = 0.$

The lines touch the cubics in the given points of contact.

For the second pair of points the complexes are

$$(x-uh^{\frac{1}{2}})\left[xy^2+uy^2h^{\frac{1}{2}}-u^2\,(fx+2gy)-u^2h^{-\frac{1}{2}}\right] = 0.$$

Cor.—Acnodal quartics cannot be resolved into complexes.

56. Double contact of the third order occurs between (C) and (D), when points of contact unite. When this happens

$$g^2 = 4fh, \quad 3ufh+1 = 0.$$

The preceding quartics unite to form the complexes

$$(x-uh^{\frac{1}{2}})(y-uf^{\frac{1}{2}})\left[xy+u\,(xf^{\frac{1}{2}}+yh^{\frac{1}{2}})-3u^2f^{\frac{1}{2}}h^{\frac{1}{2}}\right] = 0.$$

The lines do not touch the hyperbola, but their point of intersection is on it.

With these constants (C) and (D) assume the same forms as in § 7.

All other points in (C), except the points of contact with (D), are crunodes in (A).

57. Two vectors may unite, without necessary contact of (C) and (D).

The quartics (C) and (D) may be combined into the forms

$$6x^2y^2-6gxyu^2+u^4\,(g^2-fh)-u^6 = 0 \ldots\ldots\ldots\ldots (1),$$

$$2\,(fx^2-gxy+hy^2)^2+u^3\,(g^2-fh)+u = 0 \ldots\ldots\ldots\ldots (2),$$

Make (2) homogeneous in $(x : y)$ by the aid of (1); the condition

that this quadratic has equal roots is found to be

$$\{-u^2g^4+2ug^2(3ufh-1)+3-14ufh-5u^2f^2h^2\}^2 = 16u^2g^2fh(ug^2+5-ufh)^2.$$

This quartic in (ug^2), when expanded, takes the form found in § 8,

$$u^4g^8-4u^3g^6(7ufh-1)+2u^2g^4(39u^2f^2h^2-78ufh-1) -4ug^2(19u^3f^3h^3-3u^2f^2h^2+77ufh+3)+(3-ufh)^2(1+5ufh)^2=0.$$

58. The discriminant of this quartic is $(3ufh+1)^5(4ufh+1)$.

When $3ufh+1=0$, $(g^2-4fh)^2(g^2-16fh)^2=0$.

Case 1, when $g^2=4fh$, is considered in § 56.

Case 2, when $g^2=16fh$, and $3ufh+1=0$, (C) and (D) take the forms

$$(xy-3u^2f^{\frac{1}{2}}h^{\frac{1}{2}})(3xy-u^2f^{\frac{1}{2}}h^{\frac{1}{2}})-u^2(xf^{\frac{1}{2}}-yh^{\frac{1}{2}})^2=0 \quad\ldots\ldots\ldots(C),$$

$$(xy-3u^2f^{\frac{1}{2}}h^{\frac{1}{2}})(3xy-5u^2f^{\frac{1}{2}}h^{\frac{1}{2}})+u^2(xf^{\frac{1}{2}}-yh^{\frac{1}{2}})^2=0 \quad\ldots\ldots\ldots(D).$$

They touch in the points

$$xf^{\frac{1}{2}}=yh^{\frac{1}{2}}=\pm u(3fh)^{\frac{1}{2}}.$$

The corresponding quartics are resolved into conics

$$u^2\{xf^{\frac{1}{2}}+yh^{\frac{1}{2}}\mp 2u(3fh)^{\frac{1}{2}}\}^2-(xy-3u^2f^{\frac{1}{2}}h^{\frac{1}{2}})^2=0 \quad\ldots\ldots\ldots(A).$$

59. When $4ufh+1=0$, the quartic of § 57 becomes

$$(g^2-9fh)^2\left[g^2-(2\sqrt{3}\pm 1)^2fh\right].$$

Case 1, when $g^2=9fh$, and $4ufh+1=0$: (C) and (D) touch each other in the points

$$xf^{\frac{1}{2}}=yh^{\frac{1}{2}}=\pm u(2fh)^{\frac{1}{2}},$$

[and take the forms]

$$(xy-2u^2f^{\frac{1}{2}}h^{\frac{1}{2}})(3xy-2u^2f^{\frac{1}{2}}h^{\frac{1}{2}})-u^2(xf^{\frac{1}{2}}-yh^{\frac{1}{2}})^2=0\ldots\ldots\ldots(C),$$

$$(xy-2u^2f^{\frac{1}{2}}h^{\frac{1}{2}})(3xy-4u^2f^{\frac{1}{2}}h^{\frac{1}{2}})+u^2(xf^{\frac{1}{2}}-yh^{\frac{1}{2}})^2=0\ldots\ldots\ldots(D).$$

The corresponding quartics (A) intersect in the given points of contact

$$u^2(xf^{\frac{1}{2}}+yh^{\frac{1}{2}}-2\sqrt{2}uf^{\frac{1}{2}}h^{\frac{1}{2}})^2-(xy-2u^2f^{\frac{1}{2}}h^{\frac{1}{2}})^2=0\ldots\ldots\ldots(A).$$

Case 2. The other values of g^2, viz., $(2\sqrt{3}\pm 1)^2fh$, not being coincident, indicate six points of intersection of (C) and (D), which are cusps in the corresponding (A) quartics. See § 11.

60. The preceding theorems must be modified to suit the six species of binodal quartics; but as they have been examined at length when the nodes are at finite distances, it is not thought necessary to discuss them when the nodes are at infinity.

On the conditions for Folium-points in Binodal Quartics.

61. It has been stated (§ 2), that the maximum number of inflexions in a binodal quartic is four, and that there may be two folia, containing two inflexions each. It is proposed to find the conditions when two inflexions of a folium unite at a folium-point, or point of undulation.

At such a point both $\frac{dp}{dx}=0$ and $\frac{d^2p}{dx^2}=0$, if $p=\frac{dy}{dx}$.

62. By taking the previous binodal quartic system, wherein

$$u\equiv cx+dy-1,$$

and ξ, η are parameters,

$$\xi x+\eta y-1=-\frac{1}{u}(fx^2+2gxy+hy^2)+\frac{1}{u^3}x^2y^2 \quad \ldots\ldots\ldots \text{(A)};$$

and, differentiating thrice,

$$\xi+\eta p=-\frac{2}{u}\left[fx+g\,(y+xp)+hyp\right]+\frac{1}{u^2}(fx^2+2gxy+hy^2)(c+dp)$$
$$+\frac{2}{u^3}xy\,(y+xp)-\frac{3}{u^4}x^2y^2(c+dp) \quad \ldots\ldots\ldots (2);$$

$$0=-\frac{2}{u}(f+2gp+hp^2)+\frac{4}{u^2}\left[fx+g\,(y+xp)+hyp\right](c+dp)$$
$$-\frac{2}{u^3}(fx^2+2gxy+hy^2)(c+dp)^2+\frac{2}{u^3}\left[2xyp+(y+xp)^2\right]$$
$$-\frac{12}{u^4}xy\,(y+xp)(c+dp)+\frac{12}{u^5}x^2y^2\,(c+dp)^2 \quad \ldots\ldots\ldots\ldots (3);$$

$$0=\frac{6}{u^2}(f+2gp+hp^2)(c+dp)-\frac{12}{u^3}\left[fx+g\,(y+xp)+hyp\right](c+dp)^2$$
$$+\frac{6}{u^4}(fx^2+2gxy+hy^2)(c+dp)^3+\frac{12}{u^3}p\,(y+xp)$$
$$-\frac{18}{u^4}\left[2xyp+(y+xp)^2\right](c+dp)+\frac{72}{u^5}xy\,(y+xp)(c+dp)^2$$
$$-\frac{60}{u^6}x^2y^2\,(c+dp)^3 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4).$$

Multiply equation (3) by $\frac{3}{u}(c+dp)$, and add to (4); the terms involving f, g, h disappear, and the residue is the product.

$$\left[p-\frac{y}{u}(c+dp)\right]\left[1-\frac{x}{u}(c+dp)\right]\left[y+xp-\frac{2}{u}xy\,(c+dp)\right]=0.$$

The two first factors give by-solutions, the third factor the main solution.

63. When the two first factors are substituted in (3) and (4), we obtain $(y-xp)^2(y^2-fu^2)$ and $(y-xp)^2(x^2-hu^2)$.

Now $(y-xp=0)$ is not a condition; for, if it be coupled with either of these two factors, $u=cx+dy$, which is a false conclusion. The conditions $y^2-fu^2=0$, $x^2-hu^2=0$, are true conditions, and imply that the quartics degenerate into four complexes, of which the tangents at the nodes are factors.

They are inapplicable to biacnodal quartics.

64. Ex. $x=uh^{\frac{1}{2}}$, $c+dp=\frac{u}{x}$; whence $d(y-px)=1$.

The parameters are found from

$$\xi x+\eta y-1=-\frac{x}{u}(fx+2gy)=-h^{\frac{1}{2}}(fx+2gy)\ldots\ldots\ldots\text{(A)},$$

and after reducing from

$$\xi+\eta p=-\frac{x}{u}(f+2gp)=-h^{\frac{1}{2}}(f+2gp).$$

Hence $$\xi=c-(1+fh)h^{-\frac{1}{2}},\quad \eta=d-2gh^{\frac{1}{2}}.$$

The corresponding quartic (A),

$$u^4-u^3\left[x(1+fh)h^{-\frac{1}{2}}+2gh^{\frac{1}{2}}y\right]=x^2y^2-u^2(fx^2+2gxy+hy^2),$$

becomes the complex

$$(x-uh^{\frac{1}{2}})\left[u^3h^{-\frac{1}{2}}-u^2(fx+2gy)+y^2(x+uh^{\frac{1}{2}})\right]=0.$$

The line does not touch the nodal cubic, unless $g^2=2-2fh$, when the subsidiary curves (C) and (D) have quadruple contact with each other (§ 6).

The equations to three other complexes are similarly found.

65. The last factor of § 62 contains the main solution

$$y+xp-\frac{2}{u}xy(c+dp)=0.$$

When this condition is substituted in (3) and (4) of § 62, we obtain the locus of all folium-points in the binodal system (A),

$$2x^2y^2+u^2(fx^2-2gxy+hy^2)=0\ldots\ldots\ldots\ldots\ldots\text{(E)},$$

whether (u) be at a finite distance, or at infinity.

Ex.—Let $u\equiv x+y-1$ and $f=-g=h=-2$.

Assume the folium-point in the locus $x^2y^2=u^2(x+y)^2$, (E), to be $x=y=2u=\frac{2}{3}$, so that $p+1=0$.

By the aid of (E), (A) becomes $\xi+\eta=\frac{1}{2}\frac{9}{2}$; also $\xi-\eta=0$.

The corresponding quartic hyperbola

$$u^3\left[\tfrac{1}{4}\tfrac{9}{4}(x+y)-1\right]=x^2y^2+2u^2(x-y)^2,$$

when written in the form

$$(xy-4u^2)^2 = \tfrac{75}{4}u^2\,(u-\tfrac{1}{3})(u+\tfrac{8}{25}),$$

shows the tangent at the folium-point ($u=\frac{1}{3}$), and a conjugate or acu-bitangent, representing together four inflexions.

66. In the case of bicuspidal quartics $f=h=0$; and the locus of folium-points (E) degenerates into the coordinate axes and the conic

$$xy-gu^2=0 \ldots\ldots\ldots\

On a Class of Automorphic Functions. By W. Burnside. Received and Communicated November 12th, 1891.

1. *Introductory.*

In a series of memoirs published in the first, third, fourth and fifth volumes of the *Acta Mathematica*, M. Poincaré has developed the theory of discontinuous groups of linear substitutions and of the one-valued functions which are unaltered by the substitutions of such a group.

In this paper I shall adhere as closely as possible to the notation and nomenclature used by M. Poincaré, and it will be convenient to state these at once as regards the substitutions.

A substitution which changes z into $\frac{\alpha z+\beta}{\gamma z+\delta}$, written $\left(z, \frac{\alpha z+\beta}{\gamma z+\delta}\right)$, is in its normal form when $\alpha\delta-\beta\gamma=1$, and the real part of α is positive.

Two points will always be unchanged by the substitution. If these are different, it can be written in the form

$$\frac{t-a}{t-b}=K\frac{z-a}{z-b},$$

where a, b are the unchanged (or double) points of the substitution.

When K is real, the substitution is called hyperbolic; when mod K is equal to 1, it is called elliptic, and in all other cases loxodromic. If the double points coincide, the substitution is called parabolic, and can be written in the form

$$\frac{1}{t-a}=\frac{1}{z-a}+A.$$

If $\alpha-1$, β, γ, $\delta-1$ are all infinitesimal, the substitution is called an infinitesimal substitution; and a group of substitutions is called discontinuous when it contains no infinitesimal substitution. The theory of groups of a finite number of different substitutions has been completely dealt with by, among others, Prof. Cayley and Prof. F. Klein, the most detailed accounts of it being contained in the memoir "On the Schwarzian Derivative and the Polyhedral Functions" by the former, and the "Vorlesungen über das Ikosaëder" of the latter. These groups M. Poincaré leaves on one side, as also those groups the substitutions of which cannot all be derived from a finite number of fundamental substitutions.

If all the substitutions of a group preserve one circle unchanged

(for simplicity this circle can always be taken to be the axis of x), and transform the regions inside and outside this circle each into itself, M. Poincaré speaks of it as a fuchsian group; in the other case the group is called kleinian. His method of dealing with groups of substitutions is a geometrical one. He shows that, corresponding to a discontinuous group, a division of the z-plane, or part of the plane, into an infinite number of regions can be made, with the following properties. To each substitution $\left(z, \frac{\alpha_i z+\beta_i}{\gamma_i z+\delta_i}\right)$ of the group will correspond a particular region R_i of the plane, in such a way that when z is within R_0, $\frac{\alpha_i z+\beta_i}{\gamma_i z+\delta_i}$ will be within R_i. The regions R_i fill either the whole plane or the particular part of it exactly once, *i.e.*, they neither overlap each other nor leave uncovered portions. The region R_0 (called the generating polygon), and therefore also the other regions, can always be chosen so that their boundaries are arcs of circles, which, if the group is fuchsian, intersect the unchanged circle at right angles. The question as to whether the regions R_i will cover the whole plane or only a portion of it (it being understood that R_0 does not consist of two or more detached areas) will depend upon the nature of the fundamental substitutions of the group, and this difference leads to a division of the groups, and of the functions which are unchanged by them, into two classes*; namely, a first class, in which the regions R_i cover the whole plane, and the corresponding functions exist in the whole plane, and a second class, in which the functions have what is called a "natural limit." M. Poincaré considers cases of both these classes in his memoirs, but deals at considerably the greater length with the second class. Prof. Klein, in his investigations on the subject, which are contained in most detailed form in a memoir entitled "Neue Beiträge zur Riemann'schen Functionentheorie," (*Math. Ann.*, Band 21), limits himself expressly to the second class of groups.

The analytical expressions of the functions which are unaltered by the substitutions of a group are obtained in the following way by M. Poincaré. He first shows that if $\left(z, \frac{\alpha_i z+\beta_i}{\gamma_i z+\delta_i}\right)$ be any substitution of the group in its normal form, then the series

$$\Sigma \bmod (\gamma_i z+\delta_i)^{-2m},$$

* M. Poincaré distinguishes seven families of groups; of these the first, second, and sixth form what I have ventured to call the second class, while the rest make up the first class.

where m is a positive integer greater than unity, is an absolutely convergent series, except for values of z which make one or more terms of the series infinite.

It follows that the series

$$\sum_i H\left(\frac{\alpha_i z+\beta_i}{\gamma_i z+\delta_i}\right)(\gamma_i z+\delta_i)^{-2m},$$

where $H(z)$ represents a rational integral function, is uniformly convergent, except for particular values of z, and therefore that it defines a one-valued continuous function of z. Such a function M. Poincaré calls a theta-fuchsian or theta-kleinian function. Its fundamental property is shown to be the following, viz.: if

$$\Theta(z) = \sum_i H\left(\frac{\alpha_i z+\beta_i}{\gamma_i z+\delta_i}\right)(\gamma_i z+\delta_i)^{-2m},$$

then
$$\Theta\left(\frac{\alpha z+\beta}{\gamma z+\delta}\right) = (\gamma z+\delta)^{2m}\,\Theta(z),$$

$\left(z, \frac{\alpha z+\beta}{\gamma z+\delta}\right)$ being any substitution of the group.

If now $\Theta(z)$, $\Theta_1(z)$ are two such functions, formed with the same m but different functions $H(z)$, $H_1(z)$, then their ratio, if it is not a constant, will be a function which is unaltered by the substitutions of the group.

As regards the theorem that $\sum \mod(\gamma_i z+\delta_i)^{-2m}$ is a convergent series for integral values of m greater than unity, M. Poincaré does not, in his memoir on fuchsian functions, give any reasons for not dealing with the case $m=1$; but in a later memoir ("Sur les Groupes des Équations Linéaires," *Acta Math.*, Vol. IV., p. 308), he says: "Toujours dans le cas d'un groupe fuchsien, la série

$$\sum \mod(\gamma_i z+\delta_i)^{-2}$$

n'est pas convergent."

That this statement is not universally true, may be seen at once by considering the group arising from the repetition of a single hyperbolic or loxodromic substitution. This may be written in the form

$$\frac{t-a}{t-b} = K\frac{z-a}{z-b} \quad (\mod K > 1),$$

or
$$t = \frac{(bK-a)z-ab(K-1)}{(K-1)z-(aK-b)};$$

so that $$\gamma_i z+\delta_i = \frac{(K^{\frac{1}{2}i}-K^{-\frac{1}{2}i})z-(aK^{\frac{1}{2}i}-bK^{-\frac{1}{2}i})}{b-a},$$

and the series in this case is

$$(b-a)^2 \sum_{-\infty}^{\infty} \mathrm{mod}\left(\frac{1}{K^{\frac{1}{2}i}(z-a)-K^{-\frac{1}{2}i}(z-b)}\right)^2,$$

which certainly is convergent.

M. Poincaré's statement with respect to the divergency of

$$\sum_i \mathrm{mod}\,(\gamma_i z+\delta_i)^{-2},$$

then, clearly requires some limitation. I have endeavoured to show that, in the case of the first class of groups, this series is convergent, but at present I have not obtained a general proof. I shall offer two partial proofs of the convergency; one of which applies only to the case of fuchsian groups, and for that case is general, while the other will also apply to kleinian groups, but only when certain relations of inequality are satisfied. It follows, from the convergency of

$$\sum_i \mathrm{mod}\,(\gamma_i z+\delta_i)^{-2},$$

that for the first class of groups, to which all my results are limited, functions of the form

$$\sum_i H\left(\frac{\alpha_i z+\beta_i}{\gamma_i z+\delta_i}\right)(\gamma_i z+\delta_i)^{-2}$$

exist; and the existence of these functions is shown to lead to a mode of formation of fuchsian* and kleinian functions (*i.e.*, functions which are unaltered by the substitutions of the groups) which is in a certain way simpler than M. Poincaré's, and which moreover directly connects these functions with an already well-explored branch of analysis.

I shall best explain the nature of the method aimed at, and the point of view taken, by means of a particular case, which is also the one I have treated in greatest detail. Suppose, then, that the generating polygon of the group is the space outside $2n$ circles, each of which is external to all the others. The n fundamental substitutions of the group, which must be hyperbolic or loxodromic if the group is discontinuous, will make these circles correspond in pairs; and if, by bending and stretching, without tearing, the corresponding points on the pairs of circles are brought to coincidence, the polygon will be

* I have used the phrase "automorphic function," as introduced by Professor Klein, to denote generally any function which is unchanged by the substitutions of a discontinuous group, whatever be the nature of the group.

turned into a closed $(n+1)$-ply connected surface. On such a surface n everywhere-finite complex functions of position must exist, such that when a variable point describes any path on the surface, and returns to its original position, any one of these functions will only be altered by the addition of integral multiples of certain definite periods. [Considering the closed surface as a Riemann's surface extended in space, these functions are the n integrals of the first species upon it.] Also, in terms of n-ple θ-functions, with these n functions as their arguments, any rational algebraical function on the surface can be expressed.

Corresponding to these n integrals of the first species on the surface, there must be, on the z-plane, n independent functions which are everywhere finite (except at the singular points of the group, *i.e.*, the double points of its substitutions), and which increase by integral multiples of certain constant quantities when a variable point passes from a position z to any one of the corresponding positions $\frac{\alpha_i z+\beta_i}{\gamma_i z+\delta_i}$; or when it describes a closed path surrounding one or more of the circles. When once these functions are found, it must follow, from the correspondence between the z-plane and the before-constructed Riemann's surface, that any functions which are unaltered by the substitutions of the group must be capable of representation by means of multiple θ-functions, with these particular functions for their arguments. Following M. Poincaré's method, I form the everywhere-finite theta-fuchsian or theta-kleinian functions of the form

$$\Sigma H\left(\frac{\alpha_i z+\beta_i}{\gamma_i z+\delta_i}\right)(\gamma_i z+\delta_i)^{-2};$$

i.e., the functions which are only infinite at the singular points of the group; and it appears that there are n such functions, and that their integrals are the n required functions which on the z-plane have properties analogous to the n integrals of the first species on the Riemann's surface. I then show how to form functions which bear exactly the same relations to normal integrals of the second and third kinds, on the Riemann's surface, that the n functions just referred to bear to the integrals of the first kind. All the methods, then, by which uniform functions of position on the Riemann's surface are expressed in terms of the Abelian integrals on it, may be applied at once to the expression of automorphic functions by means of the three classes of functions which are the analogues of the integrals of the first, second, and third kinds.

2. *On the Convergency of* $\Sigma_i \bmod (\gamma_i z+\delta_i)^{-2}$ *in certain cases.*

First Proof.—The following proof, which is modelled on the lines of M. Poincaré's first proof of the convergency of $\Sigma_i \bmod (\gamma_i z+\delta_i)^{-4}$, applies only to the case of fuchsian groups of the first class. I suppose that the real axis is taken for the unchanged circle, and that the point $z=\infty$ is not a singular point of the group. These suppositions involve no real loss of generality. Since, by supposition, the group is of the first class, the generating polygon may be taken to enclose that part of the real axis which includes the point at infinity. Describe any circle $PQAB$, such that the area bounded by it lies entirely within the generating polygon, and suppose that it meets the real axis in the points A and B. Let $P_iQ_iA_iB_i$ be the circle into which the substitution $\left(z, \frac{\alpha_i z+\beta_i}{\gamma_i z+\delta_i}\right)$ transforms the circle $PQAB$, A_iB_i being the segment of the real axis corresponding to AB, and P_iQ_i the circular arc corresponding to PQ.

Then it follows that :—

(i.) ΣA_iB_i is finite, for it is less than that part of the real axis which lies outside the generating polygon;

(ii.) that therefore the sum of the circumferences of all the circles is finite, for they all cut the real axis at the same angle;

(iii.) and therefore that ΣP_iQ_i is finite.

The point $-\delta_i/\gamma_i$, being the homologue of the point at infinity in the substitution $\left(z, \frac{\alpha_i z+\beta_i}{\gamma_i z+\delta_i}\right)$, is a point J_i, within the polygon R_i, and lying on the real axis. When z lies on the arc PQ, the ratio of the greatest to the least value of $\bmod (\gamma_i z+\delta_i)^{-2}$ is the square of the ratio of the distances between J_i and the furthest and nearest points from it on PQ. Now the points J_i all lie on a finite portion of the axis of x; and therefore, if d_1 and d_2 be respectively the absolutely greatest and least distances of any of the points J_i from the arc PQ, the ratio of the greatest to the least value of $\bmod (\gamma_i z+\delta_i)^{-2}$, when z lies on the arc PQ, is less than $(d_1/d_2)^2$, whatever i may be.

If z is the point P, $\bmod (\gamma_i z+\delta_i)^{-2}$ is the ratio of an infinitesimal linear element at P_i to the corresponding element at P, and hence, when z is any point on the arc PQ,

$$\bmod (\gamma_i z+\delta_i)^{-2} < \left(\frac{d_1}{d_2}\right)^2 \frac{P_iQ_i}{PQ};$$

therefore $\sum_i \mod (\gamma_i z+\delta_i)^{-2} < \left(\frac{d_1}{d_2}\right)^2 \frac{1}{PQ} \Sigma P_i Q_i$;

or, since $\Sigma P_i Q_i$ is finite, the series is absolutely convergent.

Second Proof. — The proof just given applies only to fuchsian groups; depending directly, as it does, on the fact that there is a circle (or straight line) which remains unchanged by all the substitutions. I now go on to give a proof which will apply, when certain relations of inequality are satisfied, to kleinian groups similar to the one referred to in the introduction.

Suppose, then, that the generating polygon is the space outside $2n$ given circles, each of which is external to all the others. The fundamental substitutions of the group will be n substitutions, which make the circles correspond in pairs, and which, since the circles are external to each other, and the group is discontinuous, are necessarily either hyperbolic or loxodromic. The various points $-\delta_i/\gamma_i$ are all inside the given circles, and when z does not coincide with one of these points, a quantity M may be found which is less than $\mod\left(z+\frac{\delta_i}{\gamma_i}\right)$, whatever i may be.

It follows that

$$\sum_i \mod (\gamma_i z+\delta_i)^{-2} < \sum_i M^{-2} \mod \gamma_i^{-2};$$

and therefore that it is sufficient to consider the series $\sum_i \mod \gamma_i^{-2}$.

For shortness of statement I deal with the case $n = 2$, but it will be clear that the same reasoning will apply, whatever be the value of n.

Let the S, S' be the two fundamental substitutions of the group.

All the substitutions of the group may then be arranged in the following way, viz.: —

S, S', S^{-1}, S'^{-1}, 4 substitutions involving one operation;

SS, SS', SS'^{-1}: $S^{-1}S^{-1}$, $S^{-1}S'^{-1}$, $S^{-1}S'$:

$S'S'$, $S'S$, $S'S^{-1}$: $S'^{-1}S'^{-1}$, $S'^{-1}S^{-1}$, $S'^{-1}S$:

4×3 substitutions involving two operations;

then 4×3^2 substitutions involving three operations; and so on.

Now, since the fundamental substitutions are necessarily either

hyperbolic or loxodromic, they will be of the forms

$$S \equiv \left(z, \frac{\alpha z+\beta}{\gamma z+\delta}\right), \quad \gamma = \frac{K^{\frac{1}{2}}-K^{-\frac{1}{2}}}{b-a}, \quad \delta = \frac{bK^{-\frac{1}{2}}-aK^{\frac{1}{2}}}{b-a};$$

$$S' \equiv \left(z, \frac{\alpha' z+\beta'}{\gamma' z+\delta'}\right), \quad \gamma' = \frac{K'^{\frac{1}{2}}-K'^{-\frac{1}{2}}}{b'-a'}, \quad \delta' = \frac{b'K'^{-\frac{1}{2}}-a'K'^{\frac{1}{2}}}{b'-a'}.$$

Where mod K and mod K' are both greater than unity.

If $$S_n \equiv \left(z, \frac{\alpha_n z+\beta_n}{\gamma_n z+\delta_n}\right)$$

is any substitution of the n^{th} set, then

$$SS_n \equiv \left(z, \frac{\alpha_{n+1} z+\beta_{n+1}}{\gamma_{n+1} z+\delta_{n+1}}\right),$$

where $$\alpha_{n+1} = \alpha\alpha_n+\beta\gamma_n, \quad \beta_{n+1} = \alpha\beta_n+\beta\delta_n,$$

$$\gamma_{n+1} = \gamma\alpha_n+\delta\gamma_n, \quad \delta_{n+1} = \gamma\beta_n+\delta\delta_n,$$

and $$\frac{\gamma_{n+1}}{\gamma_n} = \gamma\frac{\alpha_n}{\gamma_n}+\delta = K^{\frac{1}{2}}\frac{\frac{\alpha_n}{\gamma_n}-a}{b-a}+K^{-\frac{1}{2}}\frac{b-\frac{\alpha_n}{\gamma_n}}{b-a}.$$

If SS_n is a substitution of the $\overline{n+1}^{\text{th}}$ set, the last substitution in S_n must *not* be S^{-1}; and I shall now show that this involves that $\frac{\alpha_n}{\gamma_n}$ and a are not within the same one of the four circles bounding the generating polygon.

Call the circles A, B, A', B', the four points a, b, a', b' being supposed inside the circles with the same letters. Then S changes the outside of A into the inside of B; S' changes the outside of A' into the inside of B'; and S'^{-1} changes the outside of B' into the inside of A'. Hence, if α_n/γ_n is inside A, S must be the last substitution of S_n, and (writing $S_n = SS_{n-1}$) $\alpha_{n-1}/\gamma_{n-1}$ must be inside A. The same reasoning may be repeated continually; but the point a/γ is inside B, and therefore the point α_n/γ_n cannot be inside A, and must be inside either B, A' or B'. A lower limit, independent of n, can therefore be found for the modulus of each of the fractions

$$\frac{\frac{\alpha_n}{\gamma_n}-a}{b-a}$$

which occur in the various ratios γ_{n+1}/γ_n; and, therefore, if mod K

and mod K' are sufficiently great as compared with these limits, the inequality

$$\text{mod}\,\frac{\gamma_{n+1}}{\gamma_n} > k,$$

where k is an arbitrarily chosen quantity, can be satisfied, whichever of the $\overline{n+1}^{\text{th}}$ set of the substitutions S_{n+1} may be, and also whatever n may be.

Hence, if $\bar{\gamma}$ be the greater of the two quantities γ and γ',

$$\Sigma\,\gamma_i^{-2} < \bar{\gamma}^{-2}\left[4 + \frac{4\cdot 3}{k^2} + \frac{4\cdot 3^2}{k^4} + \ldots\right];$$

or the series is convergent if

$$k^2 > 3.$$

When the double points of the fundamental substitutions are given, the series in question can clearly be made convergent by taking the moduli of the multipliers sufficiently great; but, again, when the moduli of the multipliers are sufficiently great, the group given by the substitutions is certainly a discontinuous one. Hence this second proof certainly establishes the existence of kleinian groups of the kind considered, for which $\Sigma_i \text{mod}\,(\gamma_i z + \delta_i)^{-2}$ is a convergent series, though it does not prove that the series is convergent for every such kleinian group.

If one or more of the fundamental substitutions become parabolic, in such a way that the corresponding pairs of circles touch each other externally, the preceding proof may very simply be modified to show again that there exist, in such a case, kleinian groups for which the series in question is convergent; but it does not seem to me to be possible to apply it to the cases in which the groups contain elliptic substitutions. Since, however, the first proof applies to the case of fuchsian groups containing elliptic substitutions, and, by a suitable infinitesimal change of the parameters of the fundamental substitutions, such a group will become kleinian, it would appear very probable that there must also be kleinian groups of the first class containing elliptic substitutions for which $\Sigma_i \text{mod}\,(\gamma_i z + \delta_i)^{-2}$ is convergent. By limiting the investigation of this section to the series $\Sigma\,\text{mod}\,(\gamma_i z + \delta_i)^{-2}$, I do not intend to imply that the series $\Sigma_i \text{mod}\,(\gamma_i z + \delta_i)^{-1}$ is necessarily divergent. Indeed, for the group

formed by a single hyperbolic or loxodromic substitution, it is clearly convergent; and the second proof just given, for the convergency of $\sum_i \text{mod}\,(\gamma_i z+\delta_i)^{-2}$, may evidently be altered to show the convergence of $\sum \text{mod}\,(\gamma_i z+\delta_i)^{-1}$ for certain groups derived from more than one fundamental substitution. It is not necessary, however, in the sequel, to consider functions whose existence depends on the convergency of this latter series.

3. *On the Functions $\theta(z, a)$ for Groups of the First Class.*

Having established the convergency of the series $\sum_i \text{mod}\,(\gamma_i z+\delta_i)^{-2}$ for all fuchsian and certain kleinian groups of the first class, I now propose to consider the properties of the corresponding theta-fuchsian and theta-kleinian series. In the first place, I deal with such a group as that described in the introduction; that is to say, a group derived from n fundamental, hyperbolic or loxodromic, substitutions. For such a group it follows that the series

$$\sum_i H\left(\frac{\alpha_i z+\beta_i}{\gamma_i z+\delta_i}\right)(\gamma_i z+\delta_i)^{-2}$$

is a uniformly convergent series, except for certain particular values of z, and therefore that it defines a one-valued continuous function.

It will be convenient to use a special symbol for this series in the case when $H(z)$ is a function of the first degree, and for this purpose I define $\theta(z, a)$ by the equation

$$\theta(z, a) = \sum_i \frac{(\gamma_i z+\delta_i)^{-2}}{\dfrac{\alpha_i z+\beta_i}{\gamma_i z+\delta_i} - a}.$$

If $\left(z, \dfrac{\alpha z+\beta}{\gamma z+\delta}\right)$ is any particular substitution of the group,

$$\theta\left(\frac{\alpha z+\beta}{\gamma z+\delta}, a\right) = \sum_i \frac{\left[\gamma_i(\alpha z+\beta)+\delta_i(\gamma z+\delta)\right]^{-2}}{\dfrac{\alpha_i(\alpha z+\beta)+\beta_i(\gamma z+\delta)}{\gamma_i(\alpha z+\beta)+\delta_i(\gamma z+\delta)} - a}(\gamma z+\delta)^2$$

$$= (\gamma z+\delta)^2 \sum_j \frac{(\gamma_j z+\delta_j)^{-2}}{\dfrac{\alpha_j z+\beta_j}{\gamma_j z+\delta_j} - a},$$

where
$$\left.\begin{aligned} \alpha_j &= \alpha\alpha_i+\gamma\beta_i \\ \beta_j &= \beta\alpha_i+\delta\beta_i \\ \gamma_j &= \alpha\gamma_i+\gamma\delta_i \\ \delta_j &= \beta\gamma_i+\delta\delta_i \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (A)$$

are again the coefficients of any substitution of the group.

Therefore $$\theta\left(\frac{\alpha z+\beta}{\gamma z+\delta}, a\right) = (\gamma z+\delta)^2\,\theta(z, a),$$

which is M. Poincaré's general property of all such functions.

The function $\theta(z, a)$ has two simple infinities inside any one of the regions R_i, namely, the homologues of a and ∞ that lie inside R_i. The same statement is true of $\theta\left(z, \frac{\alpha a+\beta}{\gamma a+\delta}\right)$, $\left(z, \frac{\alpha z+\beta}{\gamma z+\delta}\right)$ being again any particular substitution of the group. It follows that constant multiples of the two functions can be chosen such that their difference shall be independent of a.

Now $$\theta\left(z, \frac{\alpha a+\beta}{\gamma a+\delta}\right) = \sum_i \frac{(\gamma a+\delta)(\gamma_i z+\delta_i)^{-1}}{(\alpha_i z+\beta_i)(\gamma a+\delta)-(\gamma_i z+\delta_i)(\alpha a+\beta)};$$

put
$$\delta\alpha_i-\beta\gamma_i = \alpha_j, \quad \alpha\gamma_i-\gamma\alpha_i = \gamma_j,$$
$$\delta\beta_i-\beta\delta_i = \beta_j, \quad \alpha\delta_i-\gamma\beta_i = \delta_j,$$

so that $\alpha_j, \beta_j, \gamma_j, \delta_j$ are the coefficients of any substitution of the group, and
$$\alpha_i = \alpha\alpha_j+\beta\gamma_j, \quad \gamma_i = \gamma\alpha_j+\delta\gamma_j,$$
$$\beta_i = \alpha\beta_j+\beta\delta_j, \quad \delta_i = \gamma\beta_j+\delta\delta_j.$$

Then $$\theta\left(z, \frac{\alpha a+\beta}{\gamma a+\delta}\right) = \sum_j \frac{(\gamma a+\delta)\left[\gamma(\alpha_j z+\beta_j)+\delta(\gamma_j z+\delta_j)\right]^{-1}}{\alpha_j z+\beta_j-a(\gamma_j z+\delta_j)}$$
$$= \left(a+\frac{\delta}{\gamma}\right)\sum_j \frac{(\gamma_j z+\delta_j)^{-2}}{\left(\frac{\alpha_j z+\beta_j}{\gamma_j z+\delta_j}-a\right)\left(\frac{\alpha_j z+\beta_j}{\gamma_j z+\delta_j}+\frac{\delta}{\gamma}\right)}$$
$$= \theta(z, a)-\theta\left(z, -\frac{\delta}{\gamma}\right),$$

or
$$\theta(z, a)-\theta\left(z, \frac{\alpha a+\beta}{\gamma a+\delta}\right) = \theta\left(z, -\frac{\delta}{\gamma}\right),$$

where the right-hand side is independent of a.

The function $\theta\left(z, -\frac{\delta}{\gamma}\right)$, thus introduced, has no poles, its only singular points being the singular points of the group. For

$$\theta\left(z, -\frac{\delta}{\gamma}\right) = \sum_i \frac{\gamma}{(\gamma_i z+\delta_i)\left[\gamma(\alpha_i z+\beta_i)+\delta(\gamma_i z+\delta_i)\right]}$$

$$= \sum_i \left[\frac{\gamma\alpha_i+\delta\gamma_i}{(\gamma\alpha_i+\delta\gamma_i)z+\gamma\beta_i+\delta\delta_i} - \frac{\gamma_i}{\gamma_i z+\delta_i}\right];$$

and in this form it is clear that the residue of the function for $z = -\delta/\gamma$ is zero.

The point $-\delta/\gamma$ is one of the homologues of infinity. Denoting any one, α_i/γ_i, of these by J_i, the function $\theta(z, J_i)$ is finite everywhere, except at the singular points of the group. There are clearly an infinite number of such functions, but, as I shall next show, only n of them are linearly independent. Thus

$$\theta(z, J_p)-\theta(z, J_q) = \theta\left(z, \frac{\alpha_p}{\gamma_p}\right)-\theta\left(z, \frac{\alpha_q}{\gamma_q}\right)$$

$$= \sum \frac{\alpha_p\gamma_q-\alpha_q\gamma_p}{\left[\gamma_p(\alpha_i z+\beta_i)-\alpha_p(\gamma_i z+\delta_i)\right]\left[\gamma_q(\alpha_i z+\beta_i)-\alpha_q(\gamma_i z+\delta_i)\right]};$$

and writing, as before,

$$\delta_p\alpha_i-\beta_p\gamma_i = \alpha_j, \quad \alpha_p\gamma_i-\gamma_p\alpha_i = \gamma_j,$$
$$\delta_p\beta_i-\beta_p\delta_i = \beta_j, \quad \alpha_p\delta_i-\gamma_p\beta_i = \delta_j,$$

$$\theta(z, J_p)-\theta(z, J_q)$$

$$= -\sum_j \frac{\alpha_p\gamma_q-\alpha_q\gamma_p}{(\gamma_j z+\delta_j)\left[(\alpha_p\gamma_q-\alpha_q\gamma_p)(\alpha_j z+\beta_j)+(\beta_p\gamma_q-\delta_p\alpha_q)(\gamma_j z+\delta')\right]}$$

$$= -\sum_j \frac{(\gamma_j z+\delta_j)^{-2}}{\frac{\alpha_j z+\beta_j}{\gamma_j z+\delta_j} - \frac{\delta_p\alpha_q-\beta_p\gamma_q}{-\gamma_p\alpha_q+\alpha_p\gamma_q}}$$

$$= -\theta\left(z, \frac{\delta_p\alpha_q-\beta_p\gamma_q}{-\gamma_p\alpha_q+\alpha_p\gamma^q}\right)$$

$$= -\theta(z, J_{p^{-1}q}).$$

The same process will give

$$\theta(z, J_p) = -\theta(z, J_{p^{-1}}).$$

Hence, if S_p, S_q are two of the fundamental substitutions, and J the

homologue of infinity formed by any combination of these two, then $\theta(z, J)$ can be expressed in the form

$$m\theta(z, J_p) + n\theta(z, J_q),$$

where m and n are positive or negative integers.

It follows from this at once that if $S_1, S_2, \ldots S_n$ are the n fundamental substitutions, then, whatever homologue of infinity J may be, $\theta(z, J)$ can always be expressed in the form

$$m_1\theta(z, J_1) + m_2\theta(z, J_2) + \ldots + m_n\theta(z, J_n),$$

where $m_1, m_2, \ldots m_n$ are positive or negative integers.

Thus, as stated, not more than n of these functions are linearly independent. That the n functions written above are really independent will be proved later on.

4. *On the Integrals of the Functions* $\theta(z, J)$.

The functions $\theta(z, J)$, being everywhere finite and continuous, except at the singular points of the group, can be integrated along any path which avoids these singular points, and their integrals along any such paths of finite length will be finite.

Thus
$$\int_{z_0}^{z} \theta(z, J)\, dz = \sum_i \log \frac{\dfrac{\alpha_i z + \beta_i}{\gamma_i z + \delta_i} - J}{\dfrac{\alpha_i z_0 + \beta_i}{\gamma_i z_0 + \delta_i} - J}$$

$$= \log \prod_i \frac{\dfrac{\alpha_i z + \beta_i}{\gamma_i z + \delta_i} - J}{\dfrac{\alpha_i z_0 + \beta^i}{\gamma_i z_0 + \delta_i} - J}$$

will represent, when the path from z_0 to z is assigned (keeping clear of the singular points), a definite finite quantity, and the given path will determine the branch of the logarithm that is to be taken.

I first consider the integral when the path is confined to the generating polygon; the relation between which and the fundamental substitutions is to be taken as follows.

The generating polygon consists of the space outside n pairs of circles C_1 and C_1', C_2 and C_2', ... C_n and C_n', each external to all the others. The substitution S_p transforms the circle C_p' into the circle C_p, and the space outside the circle C_p' into the space inside the circle C_p, so that the point J_p is inside C_p, and the point J_{p-1} is inside C_p'.

An element of the integral of $\theta(z, J_p)$ may be expressed as follows:

$$\theta(z, J_p)\, dz = \sum_i d \cdot \log\left(\frac{a_i z+\beta_i}{\gamma_i z+\delta_i} - J_p\right)$$

$$= \sum_i d \cdot \log \frac{a_i\left(z - \dfrac{\delta_i J_p - \beta_i}{-\gamma_i J_p + \alpha^i}\right)}{\gamma_i\left(z+\dfrac{\delta_i}{\gamma_i}\right)}$$

$$= \sum d \cdot \log \frac{a_i (z - J_{i-1\,p})}{\gamma_i (z - J_{i-1})}.$$

When S_i is the identical substitution (z, z), the corresponding term on the right-hand side is of a slightly different form, namely, $d \cdot \log(z - J_p)$; hence

$$\theta(z, J_p)\, dz = d \cdot \log(z - J_p) + \sum_i d \cdot \log \frac{a_i (z - J_{i-1\,p})}{\gamma_i (z - J_{i-1})},$$

where now the identical substitution is not included under the sign of summation.

Now, as before stated, J_p is inside C_p, and it is easy to see that J_{i-1} and $J_{i-1\,p}$ are either both inside or both outside C_p; hence

$$\int_{C_p} \theta(z, J_p)\, dz = 2\pi i,$$

where the integral is taken in the positive direction round a closed curve within the generating polygon surrounding C_p once, and surrounding no other circle.

Again, J_p is outside C'_p, and J_{i-1}, $J_{i-1\,p}$ are either both inside or both outside C'_p, except when $i = p$, and then J_{i-1} is inside and $J_{i-1\,p}$ is outside; hence

$$\int_{C'_p} \theta(z, J_p) = -2\pi i,$$

where the integral is taken positively round C'_p.

For any other circle C_q or C'_q, J_p is outside and J_{i-1} and $J_{i-1\,p}$ are always either both inside or both outside.

Hence $$\int_{C_q} \theta(z, J_p) = \int_{C'_q} \theta(z, J_p) = 0.$$

These results prove incidentally that the n functions

$$\theta(z, J_1), \quad \theta(z, J_2) \quad \ldots\ldots \quad \theta(z, J_n)$$

are linearly independent of each other, and that they are not mere constants.

If A_p, A'_p be two corresponding points on C_p and C'_p, and B_p, B'_p any other pair of corresponding points on the same two circles, and if $A_pA'_p$, $B_pB'_p$ be joined by paths $A_pMA'_p$, $B_pNB'_p$, which do not enclose any of the circles, then

$$\int \theta(z, J)\, dz,$$

round $A_pMA'_pB'_pNB_pA_p$, vanishes, since the integrand is finite and continuous at all points within the contour. Therefore

$$\left(\int_{A_pMA'_p} - \int_{B_pNB'_p} + \int_{A'_pB'_p} - \int_{A_pB_p}\right) \theta(z, J)\, dz = 0.$$

Now, if z_1, z_2 are corresponding points on the circular arcs $A'_pB'_p$, A_pB_p,

$$z_2 = \frac{\alpha_p z_1 + \beta_p}{\gamma_p z_1 + \delta_p};$$

therefore $$\theta(z_2, J) = (\gamma_p z_1 + \delta_p)^2\, \theta(z_1, J),$$

and $$dz_2 = (\gamma_p z_1 + \delta_p)^{-2}\, dz_1;$$

therefore $$\theta(z_2, J)\, dz_2 = \theta(z_1, J)\, dz_1.$$

It follows that

$$\int_{A_pB_p} \theta(z, J)\, dz = \int_{A'_pB'_p} \theta(z, J)\, dz,$$

the integrals being taken along the circular arcs, and therefore

$$\int_{B_pNB'_p} \theta(z, J)\, dz = \int_{A_pMA'_p} \theta(z, J)\, dz.$$

Hence the integral

$$\int_{z_0}^{(\alpha_p z + \beta_p)/(\gamma_p z_0 + \delta_p)} \theta(z, J)\, dz,$$

where z_0 is any point on the circle C'_p, is independent of the particular position of the point, the path varying continuously without passing outside the fundamental polygon.

Now let the points $A_1A'_1$, $A_2A'_2$, &c. ... be joined by lines of any form, which neither cut themselves nor each other, and which do not leave the generating polygons; and regard these lines, as well as the circles, as part of the boundary of the polygon. In the figure so

the integral of $\theta(z, J)$, when the lower limit has an assigned value, is a one-valued finite continuous function.

Let $$\int_{A'_p A_p} \theta(z, J_q)\, dz = a_{qp},$$

and $$\int_{A'_p A_p} \theta(z, J_p)\, dz = a_{pp},$$

so that the quantities a_{pp}, a_{pq}, &c., are the constant values of the integrals just discussed. When the variable paths between the corresponding points are chosen so as to be reconcilable with the barriers $A_1A'_1$, $A_2A'_2$ ..., I shall show that

$$a_{pq} = a_{qp}.$$

For this purpose, let definite lower limits be chosen for the integrals (they need not yet be expressed), and write

$$\phi_p = \int^z \theta(z, J_p)\, dz.$$

Then, as stated, within the polygon with the barriers,

$$\phi_1, \phi_2 \ldots \phi_n$$

are everywhere one-valued, finite and continuous.

If the positive side of the barrier A'_pA_p is that which one would first reach by starting from any point of the circle C_p and moving round it in the direction opposite to that of the hands of a watch, then the value of ϕ_p at any point on the positive side of A'_pA_p is greater by $2\pi i$ than its value at the corresponding point on the left-hand side; while the values of φ_q at corresponding points on the right and left-hand sides are the same.

Since φ_p, ϕ_q are finite and continuous everywhere inside the polygon with barriers, it follows that

$$\int \phi_p d\phi_q,$$

extended round the complete boundary, is zero.

The complete boundary consists of the n contours made up of the n pairs of circles and the two sides of the barrier connecting each pair.

The values ϕ_p, $d\phi_q$, at corresponding points on C_r, C'_r, are given by

$$\phi_p(C_r) = a_{pr} + \phi_p(C'_r),$$
$$d\phi_q(C_r) = \quad -d\phi_q(C'_r),$$

and, at corresponding points on either side of $A'_r A_r$, ϕ_p and $d\phi_q$ have the same values. Hence the integral round the contour formed of these lines is

$$a_{qr} \int_{C'_r} d\phi_q = 0.$$

The contour formed by C_p, C'_p and $A'_p A_p$ gives

$$-2\pi i \int_{A'_p A_p} d\phi_q = -2\pi i a_{qp},$$

and the contour C_q, C'_q and $A'_q A_q$ gives

$$-a_{pq} \int_{C'_p} d\phi_q = 2\pi i a_{pq}.$$

Hence, finally, adding the various parts of the integral,

$$a_{pq} - a_{qp} = 0.$$

This proof will be seen to be the same as Prof. C. Neumann's proof of the corresponding property of Abelian integrals of the first species.

Suppose, now, that the path of integration passes outside the generating polygon, and let the homologues of the barriers be drawn for all the polygons. Then it is at once obvious that, inside any polygon with its barriers, the integrals are one-valued, finite and continuous, and also that they are continuous in crossing from one polygon to another. They are therefore one-valued and finite in the infinite plane, and continuous except at the barriers, and at corresponding points on the opposite sides of the barriers they have constant differences $\pm 2\pi i$ or zero. The value of the integrals at any point in the infinite plane, with the barriers as boundaries, may be determined as follows:—

Let O be any point in the generating polygon R_0 and O_i its homologue in R_i. It has just been shown that

$$\phi_p(O_i) - \phi_p(O),$$

is independent of the shape of the path between O and O_i, so long as it does not cut the barriers. Join OO_i by any path whatever satisfying this condition, and suppose that it passes successively through the polygons $R_0, R_a, R_b \ldots R_l, R_i$. Let O_0 be the point where it meets the boundary between R_0 and R_a; find O_a on the boundary between R_a and R_b, so that O_a is a homologue of O_0; then O_b on the boundary between R_b and R_c so that it is a homologue of O_a; and so on. Take

now $OO_0O_aO_b \ldots O_iO_i$ for the path of integration. The points O and O_i, O_0 and O_i are respectively homologues, and the paths OO_0, O_iO_i lie each entirely within one polygon. Therefore

$$\int_{OO_0} d\phi_p = \int_{O_iO_i} d\phi_p,$$

and the contributions to the whole value of the integral from OO_0 and O_iO_i destroy each other. Each of the portions O_0O_a, $O_aO_b \ldots$ is reconcilable with the homologue of one of the original barriers $A_1'A_1$, $A_2'A_2 \ldots$, taken either positively or negatively. Hence, finally, if among these homologues that of $A_1'A_1$ occurs n_1 times, that of $A_2'A_2$ n_2 times, and so on,

$$\phi_p\left(\frac{\alpha_i z+\beta_i}{\gamma_i z+\delta_i}\right) - \phi_p(z) = n_1 a_{p1} + n_2 a_{p2} + \ldots + n_n a_{pn};$$

the positive or negative integers here are definite, depending only on the particular polygon R_i; the suffix p may have any value from 1 to n.

When, finally, the barriers are dispensed with, and the now many-valued integrals are considered in the infinite plane bounded only by the singular points, any path of integration whatever can be made up by combining the path just considered with a number of closed loops. Each time such a loop cuts a barrier, $\pm 2\pi i$ or zero will be added to the value of the integral. Hence, finally, whatever be the path of integration,

$$\phi_p\left(\frac{\alpha_i z+\beta_i}{\gamma_i z+\delta_i}\right) - \phi_p(z) = 2m_p\pi i + n_1 a_{p1} + n_2 a_{p2} + \ldots + n_n a_{pn},$$

where $$p = 1, 2, \ldots n.$$

It is now clear that the functions ϕ are the analogues in the infinite plane of the integrals of the first species on the $\overline{n+1}$-ply connected Riemann's surface into which the generating (or any one) polygon can be deformed by bending and stretching till corresponding points on its surface are brought to coincide. Also, if each of the ϕ's is multiplied by $-\frac{1}{2}$, they will directly represent the n independent normal integrals of the first species; the $2n$ sections on the surface being the n curves obtained by bringing the pairs of corresponding circles to coincidence and the n closed curves into which this process transforms the barriers.

To complete the analogy, and make the introduction of the θ-functions possible, it must still be shown that the real part of

$$\Sigma n_p^2 a_{pp} + 2\Sigma n_p n_q a_{pq}$$

is essentially positive.

The proof of this, again, is essentially the same as that of the corresponding theorem with respect to Abelian integrals, and may be stated shortly.

If $$\phi_p = u_p + iv_p,$$

then, since $\Sigma n_p \phi_p$ is finite, continuous, and one-valued in the generating polygon with barriers,

$$\int \Sigma n_p u_p . \Sigma n_p \, dv_p,$$

taken in the positive direction round its boundary, is necessarily positive. The separate terms give at once

$$\int u_p dv_p = 2\pi a_{pp},$$

$$\int u_p dv_q = \int u_q dv_p = 2\pi a_{pq},$$

where a_{pp} and a_{pq} are the real parts of a_{pp} and a_{pq}; and from these equations the result follows at once.

5. *On the analogues of the Abelian integrals of the second and third kinds.*

The function $$-\Sigma_i \frac{(\gamma_i z + \delta_i)^{-2}}{\left(\frac{a_i z + \beta_i}{\gamma_i z + \delta_i} - a\right)^2}$$

has a double infinity at the point a and its homologues; while the homologues of $z = \infty$ are not infinities of the function. Its integral will therefore be a one-valued function, finite and continuous everywhere except at the point a and its homologues, and at these points it will have a simple infinity. This function will be denoted by $\psi_a(z)$, so that

$$\psi_a(z) = \Sigma_i \left(\frac{1}{\frac{a_i z + \beta_i}{\gamma_i z + \delta_i} - a} - \frac{1}{\frac{a_i z_0 + \beta_i}{\gamma_i z_0 + \delta_i} - a} \right),$$

where z_0 represents the origin of the integration, which can be chosen

at convenience. The expression in brackets can be transformed as follows:—

$$\begin{aligned}\psi_a(z) &= \sum_i \frac{z_0-z}{[a_iz+\beta_i-a(\gamma_iz+\delta_i)][a_iz_0+\beta_i-a(\gamma_iz_0+\delta_i)]}\\ &= \sum_i \frac{(-\gamma_ia+\delta_i)^{-2}(z_0-z)}{\left(\frac{\delta_ia-\beta_i}{-\gamma_ia+a_i}-z\right)\left(\frac{\delta_ia-\beta_i}{-\gamma_ia+a_i}-z_0\right)}\\ &= \sum_j (\gamma_ja+\delta_j)^{-2}\left(\frac{1}{\frac{a_ja+\beta_j}{\gamma_ja+\delta_j}-z_0}-\frac{1}{\frac{a_ja+\beta_j}{\gamma_ja+\delta_j}-z}\right)\\ &= \theta(a, z_0)-\theta(a, z).\end{aligned}$$

It follows at once that

$$\begin{aligned}\psi_a\left(\frac{a_pz+\beta_p}{\gamma_pz+\delta_p}\right)-\psi_a(z) &= \theta(a, z)-\theta\left(a, \frac{a_pz+\beta_p}{\gamma_pz+\delta_p}\right)\\ &= -\theta(a, J_p)\\ &= -\left(\frac{d\phi_p}{dz}\right)_{z=a};\end{aligned}$$

and also that

$$\psi_a\left(\frac{a_iz+\beta_i}{\gamma_iz+\delta_i}\right) = \psi_a(z)-\sum_1^n n_p\left(\frac{d\phi_p}{dz}\right)_{z=a};$$

where the integers n_p are the same as in the general formula for $\phi\left(\frac{a_iz+\beta_i}{\gamma_iz+\delta_i}\right)$. The functions $\psi(z, a)$ are thus the exact analogues of the normal integrals of the second kind on the Riemann's surface corresponding to the division of the z-plane; namely, they have one infinity inside each region, they are one-valued (*i.e.*, they do not change when the variable describes a closed path which cuts the barriers), and, when one of the substitutions of the group is performed on the variable, they increase by integral multiples of definite constants, which bear the same relation to the quasi-periods of the ϕ's that the periods of the normal integrals of the second kind bear to those of the first.

Again, the function

$$\sum_i (\gamma_iz+\delta_i)^{-2}\left(\frac{1}{\frac{a_iz+\beta_i}{\gamma_iz+\delta_i}-a}-\frac{1}{\frac{a_iz+\beta_i}{\gamma_iz+\delta_i}-b}\right)$$

is finite everywhere except at a and b and their homologues, which

points are simple infinities. It may be supposed, without loss of generality for what follows, that a and b are both within the generating polygon.

If a and b be joined by a barrier, and the homologues of this barrier be drawn, it is easy to see that the integral of the function in question will be one-valued, finite, and continuous, except at these new barriers, having values differing by $2\pi i$ at corresponding points on opposite sides of the new barriers, and, in particular, being continuous at the original barriers. If the integral from an arbitrary origin be written $\chi_{a,b}(z)$, then, in the infinite plane, as bounded by the new barriers and the singular points of the group,

$$\chi_{a,b}(z) = \sum_i \log \frac{\left(\frac{\alpha_i z+\beta_i}{\gamma_i z+\delta_i}-a\right)\left(\frac{\alpha_i z_0+\beta_i}{\gamma_i z_0+\delta_i}-b\right)}{\left(\frac{\alpha_i z+\beta_i}{\gamma_i z+\delta_i}-b\right)\left(\frac{\alpha_i z_0+\beta_i}{\gamma_i z_0+\delta_i}-a\right)},$$

where the branch of the logarithm is that which makes

$$\chi_{a,b}(z_0) = 0.$$

The expression on the right-hand side can be transformed, as in the case of $\psi(z, a)$, so as to give

$$\chi_{a,b}(z) = \sum_j \log \frac{\left(\frac{\alpha_j a+\beta_j}{\gamma_j a+\delta_j}-z\right)\left(\frac{\alpha_j b+\beta_j}{\gamma_j b+\delta_j}-z_0\right)}{\left(\frac{\alpha_j a+\beta_j}{\gamma_j a+\delta_j}-z_0\right)\left(\frac{\alpha_j b+\beta_j}{\gamma_j b+\delta_j}-z\right)}.$$

Hence
$$\chi_{a,b}\left(\frac{\alpha_p z+\beta_p}{\gamma_p z+\delta_p}\right)-\chi_{a,b}(z) = \phi_p(b)-\phi_p(a),$$

and, for any substitution,

$$\chi_{a,b}\left(\frac{\alpha_i z+\beta_i}{\gamma_i z+\delta_i}\right) = \chi_{a,b}(z)+\sum_1^n n_p\left[\phi_p(b)-\phi_p(a)\right].$$

It is also clear, by comparing the two expressions for $\chi_{a,b}(z)$, that, if the points z_1, z_2 be inside the same polygon, then

$$\chi_{a,b}(z_2)-\chi_{a,b}(z_1) = \chi_{z_2,z_1}(a)-\chi_{z_2,z_1}(b),$$

which is the equivalent of the interchange of argument and parameter for integrals of the third kind.

Just, then, as the functions $\psi(z, a)$ answer in all respects to the normal Abelian integrals of the second kind, so the functions $\chi_{a,b}(z)$ are the exact analogues of the normal integrals of the third kind.

6. *On the Functions which are unchanged by the substitutions of a group.*

It has now been shown that the functions $\phi_p(z)$, $\psi_a(z)$, $\chi_{a,b}(z)$ behave, on the z-plane, with respect to closed paths surrounding singular points of the groups and open paths passing from any point to any one of its homologues, in exactly the same way that the normal integrals of the first, second and third species on a Riemann's surface behave when the variable describes closed curves reconcilable with the a-sections and the b-sections on the surface; *i.e.*, when the variable describes any closed curve whatever on the surface. It follows therefore that the different methods of representing algebraic functions on the Riemann's surface, in terms of the normal integrals, may be applied at once to form automorphic functions on the z-plane, in terms of the functions ϕ, ψ, and χ.

Thus, if $z_1, z_2 \dots z_m$ are m points, such that no one is a homologue of any of the others (or, more simply, m different points in the same polygon), the function

$$C_0 + \sum_1^m C_r \psi_{s_r}(z)$$

will be unchanged by all the substitutions of the group if the n equations

$$\sum_1^m C_r \theta(z_r, J_p) = 0 \qquad (p = 1, 2 \dots n)$$

are satisfied. When the positions of the m given points are arbitrary, these n equations will, in general, be independent, and can therefore only be satisfied if m is equal to, or greater than, $n+1$. Hence, in general, a function which is unaltered by the substitutions of the group must take every value $n+1$ times at least in each polygon.

Again, if $f(z)$ be a function which is unaltered by the substitutions of the group, and $a_1, a_2 \dots a_m$ its zeros, $b_1, b_2 \dots b_m$ its infinities inside the generating polygon, it can be shown that

$$\sum_1^m \left[\phi_p(a_r) - \phi_p(b_r)\right] = 2m_p \pi i + \sum_{q=1}^{q=n} n'_q a_{pq}, \quad (p = 1, 2 \dots n),$$

where m'_p, n'_q are integers.

For consider the integral

$$\int \phi_p \frac{f'(z)}{f(z)} dz,$$

extended round the boundary of the generating polygon, including the

barriers. It is equal to

$$2\pi i \int_{A_p' A_p} \frac{f'(z)}{f(z)} dz - \sum_q a_{pq} \int_{C_q} \frac{f'(z)}{f(z)} dz;$$

or, since $$f(A_p) = f(A_p'),$$

and the path of the second integral is closed, the result is

$$2\pi i \left[2m_p' \pi i + \Sigma n_q' a_{pq}\right],$$

where m_p', n_q' are positive or negative integers.

In the generating polygon with barriers, ϕ_p is everywhere finite, one-valued, and continuous, and $\frac{f'(z)}{f(z)}$ is so also, except at the points $a_1, a_2 \ldots$, $b_1, b_2 \ldots b_m$, and these points are simple infinities with residues $+1$ and -1 respectively. The integral in question, then, may be evaluated again by taking it round infinitely small contours surrounding these points in the positive direction, which gives for its value

$$2\pi i \sum_1^m \left[\phi_p(a_r) - \phi_p(b_r)\right],$$

and, on equating the two values and dividing by $2\pi i$, the given result follows.

From this, the expression of an automorphic function, in terms of the functions χ, may be at once deduced. For suppose that $a_1, a_2 \ldots a_m$, $b_1, b_2 \ldots b_m$ are $2m$ points satisfying the n conditions just investigated.

Then $$\exp\left[\sum_1^m \chi_{a_r, b_r}(z) + \sum_1^n n_p' \phi_p\right],$$

is a function which in the generating polygon has the simple zeros $a_1, a_2 \ldots a_m$, and the simple infinities $b_1, b_2 \ldots b_m$; it is everywhere one-valued, and, when $\frac{\alpha_p z + \beta_p}{\gamma_p z + \delta_p}$ is written for z, it becomes multiplied by the factor

$$\exp\left\{\sum_1^m \left[\phi_p(b_r) - \phi_p(a_r)\right] + \sum_1^n n_p' a_{pq}\right\},$$

which is the same as $\exp\left[-2m_p' \pi i\right]$, or unity. It is, therefore, unchanged by all the substitutions of the group.

Finally, writing, for convenience,

$$-\tfrac{1}{2}\phi_p = u_p, \quad -\tfrac{1}{2}a_{pp} = b_{pp}, \quad -\tfrac{1}{2}a_{pq} = b_{pq},$$

and forming the θ-function

$$\theta(x_1, x_2, \ldots x_n)$$
$$= \left(\sum_{-\infty}^{\infty}\right)^n \exp\left[b_{11}n_1^2 + 2b_{12}n_1n_2 + \ldots + b_{pp}n_p^2 + 2x_1n_1 + \ldots + 2x_nn_n\right],$$

the previous results show that the conditions are satisfied for its convergence so long as the arguments x_p are finite; and if now these are replaced by the functions $u_p - c_p$, where the c_p's are any constants, the behaviour of the θ-function, when any substitution of the group is performed on the variable, will be exactly the same as that of a θ-function, whose arguments are the integrals of the first kind, on a Riemann's surface when the variable describes any closed path on the surface.

It follows that all the various known theorems, with respect to the representation of uniform functions of position on a Riemann's surface by means of θ-functions, may be now directly applied to the formation of automorphic functions in terms of the θ-functions just constructed, with the u_p's as their arguments.

7. *On the Symmetrical Case.*

When each of the n pairs of circles C_1' and C_1, C_2' and C_2, &c., which form the boundary of the generating polygon, are inverses of each other with respect to another circle C_0, and the fundamental substitutions which interchange C_1' and C_1, C_2' and C_2, &c., are all hyperbolic, M. Poincaré calls the group symmetrical. In this case the substitution S_p, which changes C_p' into C_p, is equivalent to a pair of inversions performed successively at C_p' and C_0, and any substitution of the group is equivalent to an even number of inversions with respect to the $n+1$ circles $C_0, C_1', C_2' \ldots C_n'$ (or $C_0, C_1, C_2 \ldots C_n$). Conversely, an even number of inversions with respect to any of the circles $C_0, C_1' \ldots C_n'$ (or $C_0, C_1 \ldots C_n$), is equivalent to some substitution of the group.

I shall suppose at first that the circle C_0 coincides with the real axis, and shall show further on that the results so obtained may be applied to any symmetrical group whatever.

If C_0 coincides with the axis of x, the fundamental substitutions of the group can be put in the form

$$\left(\frac{z-a_p}{z-a_p'},\ K_p\frac{z-a_p}{z-a_p'}\right),$$

where K_p is real, and a_p, a_p' are conjugate imaginaries; and it therefore follows that the substitutions of the group may be taken in pairs

$$\left(z, \frac{a_i z+\beta_i}{\gamma_i z+\delta_i}\right) \text{ and } \left(z, \frac{a_i' z+\beta_i'}{\gamma_i' z+\delta_i'}\right),$$

such that a_i, β_i, γ_i, δ_i and a_i', β_i', γ_i', δ_i' are respectively conjugate imaginaries.

If z, z' and a, a' are conjugate imaginaries, so also are

$$\frac{(\gamma_i z+\delta_i)^{-2}}{\frac{a_i z+\beta_i}{\gamma_i z+\delta_i}-a} \text{ and } \frac{(\gamma_i' z'+\delta_i')^{-2}}{\frac{a_i' z'+\beta_i'}{\gamma_i' z'+\delta_i'}-a'},$$

and therefore also $\theta(z, a)$ and $\theta(z', a')$.

Now J_p and $J_{p^{-1}}$ are in this case conjugate imaginaries, and it has been shown that in any case

$$\theta(z, J_{p^{-1}}) = -\theta(z, J_p),$$

whence it follows that

$$\theta(z, J_p) \text{ and } -\theta(z', J_p),$$

are conjugate imaginaries.

Hence, when z is real, $\theta(z, J_p)$ is a pure imaginary.

Now suppose that z' is any point on C_q', so that z is the corresponding point on C_q; then

$$\text{mod } dz = \text{mod } dz',$$

and

$$\arg dz = -\arg dz' = \phi,$$

where ϕ is the angle that the tangent at z to the circle C_q makes with the axis of x; but

$$z = \frac{a_q z'+\beta_q}{\gamma_q z'+\delta_q},$$

and therefore

$$dz = \frac{dz'}{(\gamma_q z'+\delta_q)^2};$$

and

$$(\gamma_q z'+\delta_q)^2 = e^{-2i\phi}.$$

Also, in consequence of the relation between z and z',

$$\theta(z, J_p) = (\gamma_q z'+\delta_q)^2 \theta(z', J_p)$$
$$= e^{-2i\phi} \theta(z', J_p),$$

or $$\arg\theta(z, J_p) = -2\phi + \arg\theta(z', J_p);$$

but, since $\theta(z, J_p)$ and $-\theta(z', J_p)$ are conjugate imaginaries,

$$\arg\theta(z, J_p) = \pi - \arg\theta(z', J_p);$$

therefore $$\arg\theta(z, J_p) = \frac{\pi}{2} - \phi.$$

The ratio of any two of the functions $\theta(z, J)$ is therefore real at each of the circles C'_1, C_1, &c., as well as at C_0.

The barriers in this case may be taken as straight lines perpendicular to the axis of x; and since it has been shown that, when z and z' are conjugate imaginaries, so also are $\theta(z, J_p)$ and $-\theta(z', J_p)$, it follows at once that

$$\int_{A'_Q A_Q} \theta(z, J_p)\, dz \,(= a_{pq})$$

is real, where a_{pq} is any one of the $n(n-1)$ constants.

Finally, since for points on any one of the bounding circles, including A_0,

$$\arg\theta(z, J_p) = \frac{\pi}{2} - \phi,$$

and $$\arg dz = \phi,$$

the variable part of $\int \theta(z, J_p)\, dz$, or of ϕ_p, is a pure imaginary at all the circles.

To extend these results to the case of any symmetrical group whatever, a slight digression is necessary, on the connexion between the functions of a group S_i and those of the group $\Sigma^{-1} S_i \Sigma$, where Σ is any arbitrary substitution.

If $\left(z, \frac{Az+B}{Cz+D}\right)$ be any substitution not contained in the group $\left(z, \frac{\alpha_i z+\beta_i}{\gamma_i z+\delta_i}\right)$, then the various substitutions given by

$$\left\{\frac{Az+B}{Cz+D}, \quad \frac{\alpha_i \dfrac{Az+B}{Cz+D} + \beta_i}{\gamma_i \dfrac{Az+B}{Cz+D} + \delta_i}\right\}$$

will also form a discontinuous group, that may be represented by the accented symbols

$$\left(z, \frac{\alpha'_i z+\beta'_i}{\gamma'_i z+\delta'_i}\right).$$

A division of the infinite plane into polygons, for this latter group, is obtained by transforming the polygons of the first group by the substitution

$$\left(z, \frac{Az+B}{Cz+D}\right).$$

By an algebraical process very similar to that used already in dealing with the function $\theta(z, a)$, the result of making linear substitutions for z and a in $\theta(z, a)$ is easily obtained. Thus, if

$$z = \frac{Az'+B}{Cz'+D}, \quad a = \frac{Aa'+B}{Ca'+D},$$

then, with the notation just given,

$$\theta(z, a) = (Cz'+D)^2 \sum_i (\gamma_i' z'+\delta_i')^{-2} \left(\frac{1}{\frac{\alpha_i' z'+\beta_i'}{\gamma_i' z'+\delta_i'} - a'} - \frac{1}{\frac{\alpha_i' z'+\beta_i'}{\gamma_i' z'+\delta_i'} + \frac{D}{C}} \right)$$

$$= (Cz'+D)^2 \left[\theta'(z', a') - \theta'\left(z', -\frac{D}{C}\right) \right],$$

where θ' is a function formed with the substitutions of the new group.

To the infinities $\frac{\alpha_i a+\beta_i}{\gamma_i a+\delta_i}$ and $\frac{\alpha_i}{\gamma_i}$ of $\theta(z, a)$, correspond $\frac{\alpha_i' a'+\beta_i'}{\gamma_i' a'+\delta_i'}$ and $\frac{\alpha_i' D-\beta_i' C}{\gamma_i' D-\delta_i' C}$; while the homologues of infinity in the new group are not infinities of the right-hand side.

It follows that

$$\theta(z, J_p) = (Cz'+D)^2 \theta'(z', J_p');$$

and, since

$$dz = (Cz'+D)^{-2} dz',$$

$$\varphi_p(z) = \phi_p'(z'),$$

the integrals being taken along homologous paths. In particular, the constants a_{pp}, a_{pq} for the two groups are identical.

In precisely the same way it may be shown that

$$\psi_a(z) = (Ca'+D)^2 \psi_{a'}'(z'),$$

and

$$\chi_{a,b}(z) = \chi_{a',b'}'(z').$$

If the generating polygon of the group in which C_0 is the axis of α, be transformed by any linear substitution, the new polygon

formed will be the generating polygon of a symmetrical group, the substitutions of which can be formed from those of the original one by the process just investigated; and in this manner any symmetrical polygon, and therefore the corresponding group, can be formed.

Hence for any symmetric group the constants a_{pp}, a_{pq} are real, the functions φ_p have their real parts constant at the circumferences of the $n+1$ circles, and the ratio of any two of the functions $\theta(z, J)$ is real at the circumferences of the circles.

When the symmetrical group is also fuchsian, the n pairs of circles are all cut at right angles by one circle, viz., the circle which is unchanged by the substitutions of the group.

The simplest form in which to consider the symmetrical fuchsian group is that in which the unchanged circle and A_0 are taken as two straight lines at right angles; in particular, as the axes Ox and Oy.

If S be any substitution of the group, and A represent an inversion with respect to Oy, then ASA will also be a substitution, and these two will be of the forms

$$\left(z, \frac{\alpha z+\beta}{\gamma z+\delta}\right) \quad \text{and} \quad \left(z, \frac{\alpha z-\beta}{-\gamma z+\delta}\right),$$

where α, β, γ, δ are all real; hence, pairing the substitutions in this way,

$$\theta(z, a) = \frac{1}{z-a} + \sum_i \left(\frac{(\gamma_i z+\delta_i)^{-2}}{\frac{\alpha_i z+\beta_i}{\gamma_i z+\delta_i} - a} + \frac{(-\gamma_i z+\delta_i)^{-2}}{\frac{\alpha_i z-\beta_i}{-\gamma_i z+\delta_i} - a} \right),$$

$$\theta(-z, -a) = \frac{1}{-z+a} + \sum_i \left(\frac{(-\gamma_i z+\delta_i)^{-2}}{\frac{\alpha_i z-\beta_i}{\gamma_i z-\delta_i} + a} + \frac{(\gamma_i z+\delta_i)^{-2}}{-\frac{\alpha_i z+\beta_i}{\gamma_i z+\delta_i} + a} \right);$$

whence $$\theta(-z, -a) = -\theta(z, a).$$

Now $$J_{p-1} = -J_p,$$

and $$\theta(z, J_{p-1}) = -\theta(z, J_p);$$

therefore $$\theta(-z, J_p) = \theta(z, J_p).$$

Also, since J_p is real, $\theta(z, J_p)$ and $\theta(z', J_p)$ are conjugate imaginaries, z and z' being conjugate imaginaries themselves.

Hence, if $\theta(z, J_p)$ vanishes when $z = a+ib$, it vanishes for the four values $z = \pm a \pm ib$.

Again, in this case, the function

$$\psi_a(z) - \psi_{-a}(z),$$

which is clearly not identically zero, is a fuchsian function; for

$$\begin{aligned}\psi_a\left(\frac{\alpha_p z+\beta_p}{\gamma_p z+\delta_p}\right)-\psi_{-a}\left(\frac{\alpha_p z+\beta_p}{\gamma_p z+\delta_p}\right)-[\psi_a(z)-\psi_{-a}(z)]\\ =\theta(a, J_p)-\theta(-a, J_p)\\ =0.\end{aligned}$$

Hence, in the case of symmetrical fuchsian groups, automorphic functions exist which take every value twice only in the generating polygon. It follows that the corresponding Abelian integrals are of the hyper-elliptic class.

8. *On the Zeros of* $\theta(z, J)$.

The functions $\theta(z, J)$ have no infinities, except at the singular points of the group, and hence the integral

$$\frac{1}{2\pi i}\int d\log\theta(z, J),$$

taken in the positive direction round the generating polygon, is equal to the number of simple zeros of $\theta(z, J)$ therein contained.

The barriers clearly give nothing towards the value of the integral; also, if z is a point of C_p', and z_p the corresponding point of C_p,

$$\theta(z_p, J)=(\gamma_p z+\delta_p)^2\theta(z, J);$$

and, therefore, when the integrals are taken round the circles in the directions of watch-hands, so as to make a positive circuit of the generating polygon,

$$\begin{aligned}&\int_{C_p} d\log\theta(z, J)+\int_{C_p'} d\log\theta(z, J)\\ &=\int_C\left[d\log\theta(z, J)-d\log\theta(z_p, J)\right]\\ &=-2\int_C d\log(\gamma_p z+\delta_p)=4\pi i.\end{aligned}$$

A similar result holds for each pair of circles, and the total number of zeros in the generating polygon is therefore $2n$. Of these zeros the form of the function shows that two are at infinity; for

$$\begin{aligned}\theta\left(z, \frac{\alpha_p}{\gamma_p}\right)&=\frac{1}{z-\frac{\alpha_p}{\gamma}}+\frac{(\gamma_p z+\delta_p)^{-2}}{\frac{\alpha_p z+\delta_p}{\gamma_p z+\delta_p}-\frac{\alpha_p}{\gamma_p}}+\sum_i\frac{(\gamma_i z+\delta_i)^{-2}}{\frac{\alpha_i z+\beta_i}{\gamma_i z+\delta_i}-\frac{\alpha_p}{\gamma_p}}\\ &=\frac{1}{z-\frac{\alpha_p}{\gamma_p}}-\frac{1}{z+\frac{\delta_p}{\gamma_p}}+\sum_i\frac{(\gamma_i z+\delta_i)^{-2}}{\frac{\alpha_i z+\beta_i}{\gamma_i z+\delta_i}-\frac{\alpha_p}{\gamma_p}};\end{aligned}$$

and the terms which would give a simple zero at $z = \infty$, obviously cancel each other.

The relation $$\theta\left(\frac{\alpha_i z+\beta_i}{\gamma_i z+\delta_i}, J\right) = (\gamma_i z+\delta_i)^2 \theta(z, J)$$

shows that generally, if z is a zero of $\theta(z, J)$, so also are its homologues; but that this is not true for $z = \infty$. Hence the functions $\theta(z, J)$ have, in any region R_i, $2n-2$ simple zeros, while in the generating polygon there is, in addition to these, a double zero at $z = \infty$.

The ratio of any two linear homogeneous functions of the $\theta(z, J)$'s will be an automorphic function, which generally will take every value $2n-2$ times in the generating or any other polygon. By a proper choice of the constants it is clearly possible always to form in this way a function which will take every value less than $n+1$ times, and on the other hand, the known theory of uniform functions, on a multiply-connected surface, indicates that any function which takes every value less than $n+1$ times should be capable of being represented in the way considered.

It is easy to verify that, in the case of the symmetrical fuchsian group, functions taking every value twice may be found in this way; for consider the function

$$\frac{a_2\theta(z, J_2)+a_3\theta(z, J_3)+\ldots+a_n\theta(z, J_n)}{\theta(z, J_1)}.$$

The double zeros of the numerator and denominator, at $z = \infty$, destroy each other. The $2n-2$ finite zeros of the numerator depend on the $n-2$ ratios of the constants, *i.e.*, on $2n-4$ real constants. If z_0 is a zero of the denominator, two linear relations between the $2n-4$ real constants must be satisfied in order that z_0 may also be a zero of the numerator, and, since

$$\theta(-z, J) = \theta(z, J),$$

these two relations ensure that the numerator and denominator shall have two zeros in common. Hence the available constants will enable $2n-4$ zeros of the numerator to become coincident with zeros of the denominator, so that when the constants are thus chosen, the function is one which becomes infinite (and therefore has every other value) twice only in any polygon.

The calculation that has been applied to $\theta(z, J)$ will hold similarly with any function of the form

$$\sum_i f\left(\frac{\alpha_i z+\beta_i}{\gamma_i z+\delta_i}\right)(\gamma_i z+\delta_i)^{-2},$$

where $f(z)$ is a rational function of z; so that the number of zeros of any such function in the generating polygon exceeds the number of infinities by $2n$.

9. *On Groups of the First Class which contain Elliptic Substitutions.*

I now go on to consider shortly the case in which a group contains elliptic substitutions, so that throughout this section it will be implied that the group is fuchsian, as the convergence of

$$\sum_i \text{mod}\,(\gamma_i z+\delta_i)^{-2}$$

for kleinian groups, with elliptic substitutions, has not been proved.

The essential point in which this case differs from that hitherto treated, lies in the fact that now some of the quantities

$$\theta(z, J_p) \qquad (p=1, 2 \ldots n)$$

will vanish identically.

That this must be so, if the previously given theory be correct, is obvious at once from geometrical considerations; for it is clear that the closed surface formed by bending and deforming the generating polygon till corresponding points of the boundary are brought to coincidence will no longer be $n+1$-ply connected, when some of the n fundamental substitutions are elliptic. There will, therefore, be less than n everywhere-finite integrals upon it, and, therefore, less than n ϕ-functions in connexion with the group.

Suppose, now, that an elliptic substitution S is one of the fundamental substitutions of the group; and consider the corresponding $\theta(z, J)$, defined by

$$\theta(z, J) = \sum_i \frac{(\gamma_i z+\delta_i)^{-2}}{\dfrac{\alpha_i z+\beta_i}{\gamma_i z+\delta_i} - J}.$$

The term written on the right-hand side has a simple infinity at $-\dfrac{\delta_i}{\gamma_i}$, and, if

$$\frac{(\gamma_j z+\delta_j)^{-2}}{\dfrac{\alpha_j z+\beta_j}{\gamma_j z+\delta_j} - J}$$

is the term that cancels this, then

$$\frac{\alpha_j\left(-\dfrac{\delta_i}{\gamma_i}\right)+\beta_j}{\gamma_j\left(-\dfrac{\delta_i}{\gamma_i}\right)+\delta_i} = J;$$

or $$S_i^{-1} S_j = S;$$

i.e., $$S_j = S_i S.$$

If, again, the term with suffix k is that which destroys the simple infinity $-\frac{\delta_j}{\gamma_j}$, introduced by the term with suffix j, then

$$S_k = S_j S = S_i S^2.$$

Now, if m is the period of the elliptic substitution S,

$$S^m = 1;$$

and therefore the terms in $\theta(z, J)$, corresponding to the substitutions

$$S_i, \quad S_i S, \quad S_i S^2, \ \ldots \ S_i S^{m-1},$$

are such that at any point where one of the terms becomes infinite another takes an equal and opposite infinite value.

Hence the sum of these terms, being a rational function of z which is nowhere infinite, must be a constant, and this constant is zero, as may be seen by making z infinite. It follows at once that, since $\theta(z, J)$ is in any case a uniformly convergent series, it is in this case identically zero.

[It may be interesting to give the result of a similar grouping of the terms of a $\theta(z, J)$, which corresponds to a hyperbolic substitution. The series of terms

$$\ldots S_i S^{-m}, \quad S_i S^{-m+1}, \ldots S_i, \quad S_i S, \ldots S_i S^m, \ldots$$

is then infinite, and their sum is easily shown to be equal to

$$\frac{1}{z - S_i \alpha} - \frac{1}{z - S_i \beta},$$

where α and β are the double points of the hyperbolic substitution.]

Not only will all the functions $\theta(z, J)$ which correspond to elliptic substitutions, vanish identically, but it will generally happen that some of those corresponding to hyperbolic substitutions, will also do so. For when some of the fundamental substitutions are elliptic, there will generally be certain identical relations, of the form

$$1 = S_\alpha^m S_\beta^n \ldots,$$

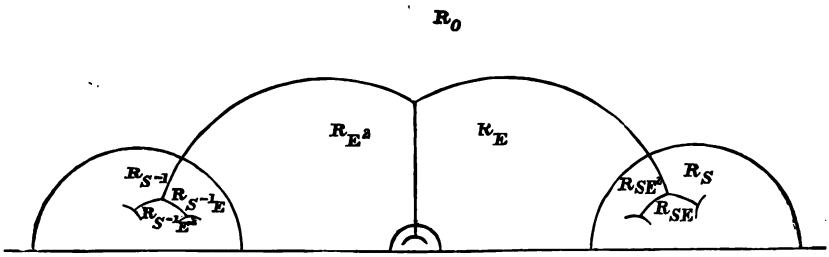
connecting them; and when these are used in the equations of the form

$$\theta(z, J_p) + \theta(z, J_{q-1}) + \theta(z, J_{p-1\,q}) = 0,$$

established in Section 3, it will be found that the result stated will follow.

Owing to the infinite variety of cases that can occur, it seems almost impossible to consider this matter generally, but the following simple example will illustrate the property in question.

Consider the fuchsian group formed from a single hyperbolic substitution S and a single elliptic substitution E of period 3. Taking the generating polygon R_0, as previously, to contain the point at infinity (and for simplicity assuming the unchanged circle to be the real axis), the division of the z-plane into regions by the group is given by the accompanying figure (the upper half only being drawn).



An inspection of the figure shows at once that SE^2 must be an elliptic substitution of period 2, or that

$$(SE^2)^2 = 1.$$

Now $$\theta(z, J_{S^{-1}}) + \theta(z, J_{E^{-1}}) + \theta(z, J_{SE^2}) = 0,$$

by the previously quoted equation; but, since E^{-1} and SE^2 are elliptic substitutions, the two latter terms of the equation are identically zero; hence, also, the first term must be. In this case, then, both the functions $\theta(z, J)$, corresponding to the fundamental substitutions, vanish; and this is in proper correspondence with the fact that the closed surface, formed by bringing corresponding points of the boundary of R_0 to coincidence, is simply-connected.

Instead of attempting to treat generally the case of a group with elliptic substitutions, I propose to continue the discussion of the special case in which all the $\theta(z, J)$'s vanish identically.

The ϕ-functions, being integrals of the $\theta(z, J)$'s, vanish identically, or rather, do not exist.

The integrals of the second kind, $\psi_a(z)$, become in this case automorphic functions, which take every value once in each polygon; and hence it immediately follows that they can all be expressed as linear functions of any one of them. It is both interesting, and will serve

as some verification of the general accuracy of the preceding investigation, to prove this result by direct calculation.

For this purpose I transform the product $\theta(a, z)\,\theta(a', z)$ in the following manner:—

$$\theta(a,z)\theta(a',z) = \sum_i \sum_j \frac{(\gamma_i a+\delta_i)^{-2}(\gamma_j a'+\delta_j)^{-2}}{\left(\frac{a_i a+\beta_i}{\gamma_i a+\delta_i}-z\right)\left(\frac{a_j a'+\beta_j}{\gamma_j a'+\delta_j}-z\right)}$$

$$= \sum_i \sum_j \frac{(\gamma_i a+\delta_i)^{-2}(\gamma_j a'+\delta_j)^{-2}}{\frac{a_i a+\beta_i}{\gamma_i a+\delta_i}-\frac{a_j a'+\beta_j}{\gamma_j a'+\delta_j}}\left(\frac{1}{\frac{a_j a'+\beta_j}{\gamma_j a'+\delta_j}-z}-\frac{1}{\frac{a_i a+\beta_i}{\gamma_i a+\delta_i}-z}\right).$$

Now
$$\sum_i \frac{(\gamma_i a+\delta_i)^{-2}}{\frac{a_i a+\beta_i}{\gamma a+\delta_i}-\frac{a_j a'+\beta_j}{\gamma_j a'+\delta_j}} = \theta\left(a, \frac{a_j a'+\beta_j}{\gamma_j a'+\delta_j}\right)$$

$$= \theta(a, a'),$$

since
$$\theta(a, J) \equiv 0;$$

hence
$$\theta(a, z)\,\theta(a', z) = \theta(a, a')\,\theta(a', z) + \theta(a', a)\,\theta(a, z).$$

In reference to this formula it is to be noticed that the zero of $\theta(a, z)$ in the generating polygon is at infinity, and therefore independent of a.

It has already been shown, in Section 5, that

$$\psi_a(z) = \theta(a, z_0) - \theta(a, z),$$

and therefore the formula just obtained is equivalent to a lineo-linear relation between any two ψ-functions. If the zero of the function be represented in the symbol by writing $\psi_a(z, z_0)$, instead of $\psi_a(z)$, the explicit relation required will be

$$\psi_{a'}(z, z_0') = \psi_{a'}(a, z_0')\frac{\psi_a(z, z_0)-\psi_a(z_0', z_0)}{\psi_a(z, z_0)-\psi_a(a', z_0)}.$$

The ψ-function in this case corresponds to what Prof. Klein calls a *fundamental function* on a simply-connected Riemann's surface; in terms of it any automorphic function, with a finite number of infinities, may be expressed rationally. Its analytical form is that of an infinite series.

But a function with precisely the same properties may be formed at once from the integrals of the third kind, *i.e.*, from the functions

$\chi_{a,b}(z)$. For, since the ϕ-functions are non-existent, it follows at once, from their previously proved properties, that

$$e^{\chi_{a,b}(z)} = \prod_i \frac{\left(\frac{a_i z+\beta_i}{\gamma_i z+\delta_i} - a\right)\left(\frac{a_i z_0+\beta_i}{\gamma_i z_0+\delta_i} - b\right)}{\left(\frac{a_i z+\beta_i}{\gamma_i z+\delta_i} - b\right)\left(\frac{a_i z_0+\beta_i}{\gamma_i z_0+\delta_i} - a\right)}$$

$$= \prod_i \frac{\left(\frac{a_i a+\beta_i}{\gamma_i a+\delta_i} - z\right)\left(\frac{a_i b+\beta_i}{\gamma_i b+\delta_i} - z_0\right)}{\left(\frac{a_i b+\beta_i}{\gamma_i b+\delta_i} - z\right)\left(\frac{a_i a+\beta_i}{\gamma_i a+\delta_i} - z_0\right)};$$

is an automorphic function with the single zero a and the single infinity b in the generating polygon. The expressions here quoted are convergent infinite products, whatever the value of z_0, the point $z = z_0$ being that at which the function takes the value unity. If then z_0 is taken infinite, the function

$$C \prod_i \frac{\frac{a_i z+\beta_i}{\gamma_i z+\delta_i} - a}{\frac{a_i z+\beta_i}{\gamma_i z+\delta_i} - b};$$

is the general expression for an automorphic function, which takes every value once in the generating polygon, in the form of an infinite product.

The passage from the infinite series to the infinite product form may be carried out as follows :—

Let $$x = \psi_a(z, b);$$

then $$\frac{dx}{dz} = \sum_i \frac{(\gamma_i z+\delta_i)^{-2}}{\left(\frac{a_i z+\beta_i}{\gamma_i z+\delta_i} - a\right)^2},$$

and $\frac{1}{x}\frac{dx}{dz}\left[= f(z)\right]$ has simple infinities at a and b, and a zero of the second order at ∞, these being its only zeros and infinities in the generating polygon; while also

$$f\left(\frac{a_i z+\beta_i}{\gamma_i z+\delta_i}\right) = (\gamma_i z+\delta_i)^2 f(z).$$

Hence it at once follows that

$$\frac{1}{x}\frac{dx}{dz} = \sum_i \left(\frac{1}{\dfrac{\alpha_i z+\beta_i}{\gamma_i z+\delta_i} - b} - \frac{1}{\dfrac{\alpha_i z+\beta_i}{\gamma_i z+\delta_i} - a} \right) (\gamma_i z+\delta_i)^{-2}$$

$$= \frac{d}{dz}\left[\chi_{b,a}(z)\right];$$

or, on integration,

$$x = \text{const.} \times e^{\chi_{b,a}(z)}$$

$$= C \prod_i \frac{\dfrac{\alpha_i z+\beta_i}{\gamma_i z+\delta_i} - b}{\dfrac{\alpha_i z+\beta_i}{\gamma_i z+\delta_i} - a}.$$

Returning for a moment to the more general case, if the closed surface formed as before by deforming the generating polygon is $n'+1$-ply connected, it is to be expected that $n-n'$ of the $\theta(z, J)$'s will vanish identically, and that, from the remaining n', a theory can be constructed in all respects similar to that of Section 4. That this is so in the case of any given group may be verified directly. If, among the fundamental substitutions of the group, there are one or more parabolic substitutions, then, in order that the group may be discontinuous, the double points of these substitutions, which are singular points of the group, must be vertices of the generating polygon. It may be shown, in a manner similar to that used for the elliptic substitutions, that the functions $\theta(z, J)$, corresponding to a parabolic substitution, vanish identically, and the preceding theory is generally applicable; except that the vertices themselves, which are the double points of the parabolic substitutions, must be reckoned as not belonging to the polygons, for, being singular points of the group, they are essentially singular points of all the functions considered.

10. *Conclusion.*

The investigations of this paper arose in an attempt to extend to the case of three or more circles the problems solved in a previous paper "On Functions determined by their Discontinuities and by a certain form of Boundary Condition," which was printed in Vol. XXII. of the Society's *Proceedings*. By using the method and notation of

that paper, a series, defined by the equation

$$w = \frac{A}{z-a} + \frac{A_0}{z-a_0} + \frac{A_1}{z-a_1} + \ldots + \frac{A_n}{z-a_n} + \frac{A_{01}}{z-a_{01}} + \frac{A_{02}}{z-a_{02}} + \ldots + \frac{A_{012}}{z-a_{012}} + \ldots + \ldots,$$

may be formed, which, if convergent, will represent in the space external to $n+1$ circles, each of which is external to all the others, a function with a single infinity at a, and whose imaginary part has constant values at the circles.

If $\left(z, \frac{\alpha_1 z+\beta_1}{\gamma_1 z+\delta_1}\right) \ldots \left(z, \frac{\alpha_n z+\beta_n}{\gamma_n z+\delta_n}\right)$ are the n linear substitutions equivalent to pairs of inversions at the circles 0 and 1, 0 and 2, ... 0 and n, respectively, and $\left(z, \frac{\alpha_i z+\beta_i}{\gamma_i z+\delta_i}\right)$ any substitution of the group formed from these as fundamental substitutions, it may be directly verified that the above series is

$$w = A\sum_i \frac{(\gamma_i a+\delta_i)^{-2}}{z-\frac{\alpha_i a+\beta_i}{\gamma_i a+\delta_i}} + A_0\sum_i \frac{(\gamma_i a_0+\delta_i)^{-2}}{z-\frac{\alpha_i a_0+\beta_i}{\gamma_i a_0+\delta_i}};$$

and that it is therefore a convergent series, and does in fact represent the function in question. An indirect proof of the accuracy of this statement will be now given, so that the direct one, which, though simple, cannot be made very short, may be omitted.

Consider, in the first case of the symmetrical group, namely, that for which the $(n+1)^{\text{th}}$ circle is taken as the real axis, the function

$$A\psi_a(z) + A'\psi_{a'}(z),$$

where A, A', and a, a' are conjugate imaginaries.

Regarded as a function of z, this expression has entirely real coefficients, and therefore will take conjugate imaginary values when z does.

Now let z be any point on the circle C_p, so that z' is the corresponding point on $C_{p'}$. Then, if

$$A\psi_a(z) + A'\psi_{a'}(z) = P+iQ,$$

$$A\psi_a(z') + A'\psi_{a'}(z') = P-iQ;$$

but

$$z = \frac{\alpha_p z'+\beta_p}{\gamma_p z'+\delta_p},$$

and therefore

$$A\psi_a(z)+A'\psi_{a'}(z)-\left[A\psi_a(z')+A'\psi_{a'}(z')\right]=A\theta(a, J_p)+A'\theta(a', J_p),$$

or
$$iQ=\tfrac{1}{2}A\theta(a, J_p)+\tfrac{1}{2}A'\theta(a', J_p).$$

It follows that, at each separate circular bounding curve of the generating polygon, the imaginary part of

$$A\psi_a(z)+A'\psi_{a'}(z)$$

has constant values.

If, now, the transformation described in Section 7 be applied to this function, it becomes

$$A(\gamma b+\delta)^2\psi_b(z)+A'(\gamma b_0+\delta)^2\psi_{b_0}(z),$$

where the ψ-functions are formed with the substitutions of the new group, while b, b_0 are inverse points with respect to the $(n+1)^{\text{th}}$ circle, which itself is the result of transforming the real axis by the substitution $\left(z, \frac{\alpha z+\beta}{\gamma z+\delta}\right)$ of the transformation.

The relations between b, b_0 and a, a' are

$$a=\frac{ab+\beta}{\gamma b+\delta}, \quad a'=\frac{ab_0+\beta}{\gamma b_0+\delta};$$

or, since a, a' are conjugate imaginaries,

$$\frac{ab_0+\beta}{\gamma b_0+\delta}=\frac{a'b'+\beta'}{\gamma' b'+\delta'};$$

and therefore
$$\frac{db_0}{(\gamma b_0+\delta)^2}=\frac{db'}{(\gamma' b'+\delta')^2}.$$

Finally, if
$$A(\gamma b+\delta)^2=B,$$

then
$$A'(\gamma b_0+\delta)^2=B'\left(\frac{\gamma b_0+\delta}{\gamma' b'+\delta'}\right)^2=B'\frac{db_0}{db'};$$

and therefore, for any symmetric group,

$$B\psi_b(z)+B'\frac{db_0}{db'}\psi_{b_0}(z)$$

is a function whose imaginary part has constant values at $n+1$ given circles (each external to all the others) and which has a single infinity, with given residue B, in the space bounded by them.

On substituting for the ψ-functions by means of the equation

$$\psi_b(z) = \theta(b, z_0) - \theta(b, z),$$

and re-writing a and a_0 for b and b_0, the expression just obtained is at once seen to differ only by a constant from the previous function w. A reference to p. 352 of my former paper, already quoted, will show that $A_0 = A' \frac{da_0}{da}$, which is the necessary relation between the coefficients.

If the point a is at infinity, so that a_0 is the centre of the $(n+1)^{\text{th}}$ circle,

$$w = A\psi_\infty(z) + A' r_0^2 \psi_{a_0}(z),$$

or, written at length,

$$w = \sum_i \left(A \left\{ \frac{a_i z + \beta_i}{\gamma_i z + \delta_i} - \frac{a_i z_0 + \beta_i}{\gamma_i z_0 + \delta_i} \right\} + A' r_0^2 \left\{ \frac{1}{\frac{a_i z + \beta_i}{\gamma_i z + \delta_i} - a_0} - \frac{1}{\frac{a_i z_0 + \beta_i}{\gamma_i z_0 + \delta_i} - a_0} \right\} \right);$$

and the real and imaginary parts of w will be respectively the velocity-potential and stream-function for a uniform streaming motion past the $n+1$ circles, the speed of the stream at an infinite distance, and the angle it makes with the real axis, being mod. A and arg. A.

The n functions $\phi_1, \phi_2 \ldots \phi_n$, the group being still considered symmetrical, are the functions of z whose imaginary and real parts give the velocity-potentials and the stream-functions for the n independent circulating motions that can take place about the $n+1$ circles, on the supposition that the circulation in any circuit enclosing all the $n+1$ circles is zero.

For, if $\kappa_1, \kappa_2 \ldots \kappa_{n+1}$ are the circulation constants for the $n+1$ circles, connected by the relation

$$\kappa_1 + \kappa_2 + \ldots + \kappa_{n+1} = 0,$$

then the function $\kappa_1\phi_1 + \kappa_2\phi_2 + \ldots + \kappa_n\phi_n$

is everywhere finite in the space external to the $n+1$ circles, while its real part is constant at each circle, and increases by κ_r when the variable describes a closed path which surrounds the r^{th} circle once. But these are the conditions that $u+iv$ should satisfy if u is the stream-function and $-v$ the velocity-potential of the proposed circulating motion.

The two examples just given will serve to show that the theory of automorphic functions of the first class may serve to elucidate considerably many two-dimensional physical problems. With respect to

the general problem dealt with in the previously quoted paper, viz., the formation of a function w with given infinities, such that the imaginary part of $we^{i\theta_r}$ is constant at the circumference of the circle C_r ($r = 1, 2, \ldots n+1$), it will be found that, even when the differences $\theta_r - \theta_s$ are commensurable with π, the terms in the infinite series for w which contain the same exponential do not arise from one of them by the operations of any sub-group of the original group; and therefore, that *w cannot* be represented as the sum of a finite number of ψ-functions formed from the substitutions of sub-groups in a manner analogous to that proved to be possible when the group arose from a single fundamental substitution.

Note on the Motion of a Fluid Ellipsoid under its own Attraction.

By M. J. M. Hill, M.A., Sc.D., Professor of Mathematics at University College, London. Communicated in Abstract, June 11th, 1891. Received January 27th, 1892.

1. In regard to this case of Fluid Motion, it was shown by Dirichlet that the particles which, at any instant, lie on an ellipsoid concentric, similar, and similarly situated to the external free surface, always do so; or, one family of surfaces, which always contain the same particles, is a family of ellipsoids which move so as always to be similar and similarly situated to the boundary.

The following additional results regarding this case of Fluid Motion were recently communicated to me by Mr. A. E. H. Love.

2. The particles which, at any instant, lie on a tangent plane to one of the ellipsoids concentric, similar, and similarly situated to the free surface, always lie on a tangent plane to this moving ellipsoid, and the same particle is always at the point of contact.

3. The enveloping cylinders of the ellipsoids mentioned above, whose generators are parallel to the vortex lines, always contain the same particles. Hence there are two families of surfaces, viz., the ellipsoids and the enveloping cylinders, which move so as to contain the same particles, and are such that one particular member of one always touches one particular member of the other.

4. The planes which are conjugate to the directions of the vortex lines with regard to the ellipsoids always contain the same particles.

I.—The object of this note is to express the component velocities in Clebsch's forms, viz.:

$$\frac{\partial \chi}{\partial x}+\lambda\frac{\partial \psi}{\partial x},\quad \frac{\partial \chi}{\partial y}+\lambda\frac{\partial \psi}{\partial y},\quad \frac{\partial \chi}{\partial z}+\lambda\frac{\partial \psi}{\partial z},$$

where $\lambda = \text{const.}$, $\psi = \text{const.}$ are vortex sheets; and to determine χ, λ, ψ as far as possible.

The following equations are taken from Basset's *Hydrodynamics*, Vol. II., pages 96, 97.

$$\omega_1 = \Omega_1+\xi,\quad \omega_2 = \Omega_2+\eta,\quad \omega_3 = \Omega_3+\zeta.$$

The kinematical condition at the surface is

$$\frac{\partial F}{\partial t}+U\frac{\partial F}{\partial x}+V\frac{\partial F}{\partial y}+W\frac{\partial F}{\partial z}=0,$$

where

$$F=(x/a)^2+(y/b)^2+(z/c)^2-1=0,$$

$$U=u+\omega_3 y-\omega_2 z,$$

$$V=v+\omega_1 z-\omega_3 x,$$

$$W=w+\omega_2 x-\omega_1 y;$$

$$u=\frac{\dot a}{a}x+\frac{\omega_3(a^2-b^2)-2a^2\zeta}{a^2+b^2}y+\frac{\omega_2(c^2-a^2)+2a^2\eta}{c^2+a^2}z,$$

$$v=\frac{\dot b}{b}y+\frac{\omega_1(b^2-c^2)-2b^2\xi}{b^2+c^2}z+\frac{\omega_3(a^2-b^2)+2b^2\zeta}{a^2+b^2}x,$$

$$w=\frac{\dot c}{c}z+\frac{\omega_2(c^2-a^2)-2c^2\eta}{c^2+a^2}x+\frac{\omega_1(b^2-c^2)+2c^2\xi}{b^2+c^2}y.$$

The condition of continuity requires that

$$\dot a/a+\dot b/b+\dot c/c=0.$$

Also the following equations hold:—

$$\frac{d}{dt}(\xi/a)-\frac{2ab\Omega_3}{a^2+b^2}(\eta/b)+\frac{2ca\Omega_2}{c^2+a^2}(\zeta/c)=0,$$

$$\frac{d}{dt}(\eta/b)-\frac{2bc\Omega_1}{b^2+c^2}(\zeta/c)+\frac{2ab\Omega_3}{a^2+b^2}(\xi/a)=0,$$

$$\frac{d}{dt}(\zeta/c)-\frac{2ca\Omega_2}{c^2+a^2}(\xi/a)+\frac{2bc\Omega_1}{b^2+c^2}(\eta/b)=0.$$

Then I find, by means of a somewhat complicated analysis, the following values of χ, λ, ψ :—

$$\chi = \tfrac{1}{2}\Big[\frac{\dot a}{a}x^2+\frac{\dot b}{b}y^2+\frac{\dot c}{c}z^2 + \frac{2\Omega_1(b^2-c^2)}{b^2+c^2}yz+\frac{2\Omega_2(c^2-a^2)}{c^2+a^2}zx+\frac{2\Omega_3(a^2-b^2)}{a^2+b^2}xy\Big]$$

$$+ \text{an arbitrary function of the time};$$

$$\lambda = \Big(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\Big)\Big(\frac{\xi^2}{a^2}+\frac{\eta^2}{b^2}+\frac{\zeta^2}{c^2}\Big)-\Big(\frac{x\xi}{a^2}+\frac{y\eta}{b^2}+\frac{z\zeta}{c^2}\Big)^2;$$

$$\psi = -\frac{abc}{\Big(\frac{\xi^2}{a^2}+\frac{\eta^2}{b^2}+\frac{\zeta^2}{c^2}\Big)^{\frac{1}{2}}}\tan^{-1}\left\{\frac{\frac{z}{c}\Big(\frac{\xi^2}{a^2}+\frac{\eta^2}{b^2}+\frac{\zeta^2}{c^2}\Big)-\frac{\zeta}{c}\Big(\frac{x\xi}{a^2}+\frac{y\eta}{b^2}+\frac{z\zeta}{c^2}\Big)}{\Big(\frac{\xi^2}{a^2}+\frac{\eta^2}{b^2}+\frac{\zeta^2}{c^2}\Big)^{\frac{1}{2}}\Big(\frac{x}{a}\frac{\eta}{b}-\frac{y}{b}\frac{\xi}{a}\Big)}\right\}$$

$$+abc\int\Big(\frac{\xi^2}{a^2}+\frac{\eta^2}{b^2}\Big)^{-1}\Big(\frac{2bc\Omega_1}{b^2+c^2}\frac{\xi}{a}+\frac{2ca\Omega_2}{c^2+a^2}\frac{\eta}{b}\Big)\,dt.$$

It is interesting to observe

(α) that the surfaces ψ = const. are a family of planes;

(β) that χ is the velocity-potential of the motion due to the motions of the enveloping case (supposed to be annihilated after the motion is set up), which have been described by Professor Greenhill in the *Proceedings of the Cambridge Philosophical Society*, Vol. IV., p. 4, so that

$$\lambda\frac{\partial\psi}{\partial x}=z\eta-y\zeta,\quad \lambda\frac{\partial\psi}{\partial y}=x\zeta-z\xi,\quad \lambda\frac{\partial\psi}{\partial z}=y\xi-x\eta.$$

The arbitrary function of the time is determined when the pressure is known at a given instant.

The values of χ, λ, ψ are not fully known until ξ, η, ζ have been determined as functions of the time.

This has been accomplished in the cases mentioned in Basset's *Hydrodynamics*, Vol. II., Cap. XV., and also

(α) when the ellipsoid becomes an elliptic cylinder, the solution being due to Lipschitz (see *Crelle-Borchardt*, LXXVIII., pp. 245–272); and

(β) when $\Omega_1=\Omega_2=\Omega_3=0$, and $\xi=\eta=0$, but ζ is a function of the time, the solution being due to Padova (see *Annali della R. Scuola Normale Superiore di Pisa—Scienze Fisiche e Mathematiche*, Vol. I., Pisa, 1871).

II.—It does not seem worth while to give the analysis by which the values of χ, λ, ψ were originally obtained, as it is so much simpler to verify them as follows:—

Putting $$\frac{D}{dt} = \frac{\partial}{\partial t} + U\frac{\partial}{\partial x} + V\frac{\partial}{\partial y} + W\frac{\partial}{\partial z};$$

then, since

$$U = \frac{\dot{a}x}{a} + \frac{2a^2(\omega_3 - \zeta)}{a^2+b^2}y - \frac{2a^2(\omega_2 - \eta)}{c^2+a^2}z,$$

it follows that

$$\frac{Dx}{dt} = U = \frac{\dot{a}}{a}x + \frac{2a^2\Omega_3}{a^2+b^2}y - \frac{2a^2\Omega_2}{c^2+a^2}z;$$

therefo $$\frac{D}{dt}(x/a) - \frac{2ab\Omega_3}{a^2+b^2}(y/b) + \frac{2ac\Omega_2}{c^2+a^2}(z/c) = 0,$$

with two similar equations.

Now, putting $$N_1 = 2bc\Omega_1/(b^2+c^2),$$
$$N_2 = 2ca\Omega_2/(c^2+a^2),$$
$$N_3 = 2ab\Omega_3/(a^2+b^2),$$

the above equations are

$$\frac{D}{dt}(x/a) - N_3(y/b) + N_2(z/c) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (A),$$

and two similar equations.

It may be observed that ξ/a, η/b, ζ/c, also satisfy the equations (A). Equations of the form (A) have been discussed by Riemann (see his *Mathematische Werke*, pp. 175–176).

They possess the property that, if X_1, Y_1, Z_1 and X_2, Y_2, Z_2 be two systems of integrals (not necessarily distinct), then

$$\frac{D}{dt}(X_1X_2 + Y_1Y_2 + Z_1Z_2) = 0.$$

Further

$$\frac{D}{dt}(Y_1Z_2 - Y_2Z_1) - N_3(Z_1X_2 - Z_2X_1) + N_2(X_1Y_2 - X_2Y_1) = 0,$$

and two similar equations.

Hence $Y_1Z_2-Y_2Z_1$, $Z_1X_2-Z_2X_1$, $X_1Y_2-X_2Y_1$ are also integrals of the equations (A).

Hence (1) $\frac{D\lambda}{dt}=0$; for,

$$x/a,\quad y/b,\quad z/c,$$
$$\xi/a,\quad \eta/b,\quad \zeta/c$$

being two systems of integrals of equations (A),

$$\frac{D}{dt}\left[(x/a)^2+(y/b)^2+(z/c)^2\right]=0,$$

$$\frac{D}{dt}\left[(\xi/a)^2+(\eta/b)^2+(\zeta/c)^2\right]=0,$$

$$\frac{D}{dt}\left[(x\xi/a^2)+(y\eta/b^2)+(z\zeta/c^2)\right]=0.$$

Now $$\lambda=\left[(x/a)^2+(y/b)^2+(z/c)^2\right]\left[(\xi/a)^2+(\eta/b)^2+(\zeta/c)^2\right]-\left[(x\xi/a^2)+(y\eta/b^2)+(z\zeta/c^2)\right]^2.$$

Hence $\frac{D\lambda}{dt}=0$.

(2) Putting
$$q_1=(y\zeta-z\eta)/(bc),$$
$$q_2=(z\xi-x\zeta)/(ca),$$
$$q_3=(x\eta-y\xi)/(ab),$$

it follows that q_1, q_2, q_3 are integrals of the equations (A).

The following algebraic identity is useful:—

$$\left[(\xi/a)^2+(\eta/b)^2+(\zeta/c)^2\right]q_3^2+\left[q_2\xi/a-q_1\eta/b\right]^2=\left[(\xi/a)^2+(\eta/b)^2\right](q_1^2+q_2^2+q_3^2).$$

(3) $$\frac{D\psi}{dt}=0;$$

for $$\psi=abc\int(N_1\xi/a+N_2\eta/b)\left\{(\xi/a)^2+(\eta/b)^2\right\}^{-1}dt$$
$$-abc\left[(\xi/a)^2+(\eta/b)^2+(\zeta/c)^2\right]^{-\frac{1}{2}}\times\tan^{-1}\left\{(q_3\xi/a-q_1\eta/b)/\left[q_3\left[(\xi/a)^2+(\eta/b)^2+(\zeta/c)^2\right]^{\frac{1}{2}}\right]\right\}.$$

Putting $$Q_1=q_3\eta/b-q_2\zeta/c,$$

and using two similar expressions for Q_2, Q_3, then Q_1, Q_2, Q_4 are integrals of the equations (A); therefore

$$\frac{DQ_3}{dt} = N_2 Q_1 - N_1 Q_2.$$

Also
$$\frac{Dq_3}{dt} = N_2 q_1 - N_1 q_2.$$

Then
$$\frac{D\psi}{dt} = abc\,(N_1\xi/a + N_2\eta/b)\left[(\xi/a)^2+(\eta/b)^2\right]^{-1}$$
$$-abc\,\frac{q_3\,(N_2Q_1-N_1Q_2)-Q_3\,(N_2q_1-N_1q_2)}{q_3^2\left\{(\xi/a)^2+(\eta/b)^2+(\zeta/c)^2\right\}+(q_2\xi/a-q_1\eta/b)^2}.$$

The denominator of the last fraction is $(q_1^2+q_2^2+q_3^2)\left\{(\xi/a)^2+(\eta/b)^2\right\}$.

The numerator is

$$\begin{aligned}
&= q_3\,(N_2q_3\eta/b - N_2q_3\zeta/c - N_1q_1\zeta/c + N_1q_3\xi/a)\\
&\quad -q_3\,(N_2q_1-N_1q_2)\,\xi/a + q_1\,(N_2q_1-N_1q_2)\,\eta/b\\
&= N_1\,(-q_1q_3\zeta/c + q_3^2\xi/a + q_2^2\xi/a - q_1q_2\eta/b)\\
&\quad +N_2\,(q_3^2\eta/b - q_2q_3\zeta/c - q_1q_3\xi/a + q_1^2\eta/b)\\
&= N_1\left[(q_1^2+q_2^2+q_3^2)\,\xi/a - q_1\,(q_1\xi/a+q_2\eta/b+q_3\zeta/c)\right]\\
&\quad +N_2\left[(q_1^2+q_2^2+q_3^2)\,\eta/b - q_2\,(q_1\xi/a+q_2\eta/b+q_3\zeta/c)\right]\\
&= (q_1^2+q_2^2+q_3^2)(N_1\xi/a+N_2\eta/b);
\end{aligned}$$

because
$$q_1\xi/a + q_2\eta/b + q_3\zeta/c = 0.$$

Substituting these values, it follows that

$$\frac{D\psi}{dt} = 0.$$

4. The expressions for the component velocities may be verified.

$$u = \dot{a}x/a + \frac{\omega_3\,(a^2-b^2)-2a^2\zeta}{a^2+b^2}\,y + \frac{\omega_2\,(c^2-a^2)+2a^2\eta}{c^2+a^2}\,z,$$

$$\frac{\partial\chi}{\partial x} = \dot{a}x/a + \frac{\Omega_3\,(a^2-b^2)}{a^2+b^2}\,y + \frac{\Omega_2\,(c^2-a^2)}{c^2+a^2}\,z;$$

therefore

$$u-\frac{\partial\chi}{\partial x}=\frac{y}{a^2+b^2}\left\{(a^2-b^2)(\omega_3-\Omega_3)-2a^2\zeta\right\}$$
$$+\frac{z}{c^2+a^2}\left\{(c^2-a^2)(\omega_2-\Omega_2)+2a^2\eta\right\}$$
$$=z\eta-y\zeta,$$

$$\frac{\partial\psi}{\partial x}=-abc\left[\left\{(\xi/a)^2+(\eta/b)^2\right\}(y\zeta-z\eta)/abc\right]$$
$$\div\left[q_3^2\left\{(\xi/a)^2+(\eta/b)^2+(\zeta/c)^2\right\}+\left\{q_3\xi/a-q_1\eta/b\right\}^2\right]$$
$$=-\left[\left\{(\xi/a)^2+(\eta/b)^2\right\}(y\zeta-z\eta)\right]/\left[(q_1^2+q_2^2+q_3^2)\left\{(\xi/a)^2+(\eta/b)^2\right\}\right]$$
$$=(z\eta-y\zeta)/(q_1^2+q_2^2+q_3^2)$$
$$=(z\eta-y\zeta)/\lambda;$$

therefore
$$u-\frac{\partial\chi}{\partial x}=\lambda\frac{\partial\psi}{\partial x}.$$

Hence
$$u=\frac{\partial\chi}{\partial x}+\lambda\frac{\partial\psi}{\partial x},$$

and two similar equations.

III. To obtain demonstrations of the results which stand at the head of this note, the properties of the equations (A) must be employed.

(i.) Observing that these equations are satisfied by x/a, y/b, z/c, it follows that

$$(x/a)^2+(y/b)^2+(z/c)^2=\text{const.}$$

(This is Dirichlet's theorem.)

(ii.) If x_1, y_1, z_1 be the coordinates of a particle on the surface

$$(x/a)^2+(y/b)^2+(z/c)^2=C,$$

and if x, y, z be the coordinates of a particle on the tangent plane to this surface, then x/a, y/b, z/c, and also x_1/a, y_1/b, z_1/c satisfy the equations (A); therefore

$$\frac{D}{dt}\left[xx_1/a^2+yy_1/b^2+zz_1/c^2\right]=0.$$

Hence the particle x, y, z always remains on the tangent plane to the surface

$$(x/a)^2+(y/b)^2+(z/c)^2 = C.$$

This is theorem (2).

(iii.) The cylinder, whose generators are parallel to the vortex lines, enveloping the ellipsoid

$$(x/a)^2+(y/b)^2+(z/c)^2 = C,$$

is $$\left[(\xi/a)^2+(\eta/b)^2+(\zeta/c)^2\right]\left[(x/a)^2+(y/b)^2+(z/c)^2-C\right] - \left[x\xi/a^2+y\eta/b^2+z\zeta/c^2\right]^2 = 0,$$

i.e., $$\lambda - C\left[(\xi/a)^2+(\eta/b)^2+(\zeta/c)^2\right] = 0.$$

But $$\frac{D\lambda}{dt} = 0, \quad \frac{D}{dt}\left[(\xi/a)^2+(\eta/b)^2+(\zeta/c)^2\right] = 0;$$

therefore

$$\frac{D}{dt}\left[\left\{(\xi/a)^2+(\eta/b)^2+(\zeta/c)^2\right\}\left\{(x/a)^2+(y/b)^2+(z/c)^2-C\right\} - \left\{x\xi/a^2+y\eta/b^2+z\zeta/c^2\right\}^2\right] = 0.$$

Hence the cylinders, whose generators are parallel to the vortex lines, and which envelope the ellipsoids

$$(x/a)^2+(y/b)^2+(z/c)^2 = \text{const.},$$

contain the same particles of fluid.

This is theorem (3).

(iv.) The planes conjugate to the direction of the vortex lines are

$$x\xi/a^2+y\eta/b+z\zeta/c^2 = \text{const.}$$

ut $$\frac{D}{dt}\left[x\xi/a^2+y\eta/b^2+z\zeta/c^2\right] = 0.$$

Hence these planes contain the same particles of fluid.

This is theorem (4).

Note on Finding the G points of a given Circle with respect to a given Triangle of Reference. *By* JOHN GRIFFITHS, M.A.

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The problem of finding the G points of a given circle, with regard to a given triangle, may be solved by means of the results obtained in Section 11 of my "Notes on the Recent Geometry of the Triangle."

If the given triangle ABC, whose sides are a, b, c, be taken as the triangle of reference, and x, y, z satisfy the relation

$$ax+by+cz = ayz+bzx+cxy,$$

the equation of a circle can be written in the form

$$\lambda x+\mu y+\nu z = 1.$$

Again, a G point of a circle is a focus, say G, of a conic which touches the sides of the given triangle of reference, and has double contact with the circle, the chord of contact being parallel to the line joining $G\ (x, y, z)$ to the inverse point $\left(\frac{1}{x}, \frac{1}{y}, \frac{1}{z}\right)$.

This being so, it will be seen that for a given triangle a given circle has, in general, four pairs of G points, since four conics can be drawn to satisfy the conditions in question. In other words, a given circle belongs to four different systems of G circles, if a G system of circles be defined to be one in which every circle of the system has double contact with a conic touching the sides of the triangle of reference.

If, then, we take $\lambda x+\mu y+\nu z = 1$

for the equation of the given circle, and write

$$f = \sqrt{(b\nu+c\mu-a)^2-4bc\mu\nu},$$

$$g = \sqrt{(c\lambda+a\nu-b)^2-4ca\nu\lambda},$$

$$h = \sqrt{(a\mu+b\lambda-c)^2-4ab\lambda\mu},$$

the isogonal coordinates x, y, z of the four pairs of G points of the

circle are

$$\left.\begin{aligned} x &= \frac{b\lambda - a\mu + c + h}{c\lambda - a\nu + b + g}, \\ y &= \frac{c\mu - b\nu + a + f}{a\mu - b\lambda + c + h}, \\ z &= \frac{a\nu - c\lambda + b + g}{b\nu - c\mu + a + f}, \end{aligned}\right\} \qquad \left.\begin{aligned} x^{-1} &= \frac{b\lambda - a\mu + c + h}{c\lambda - a\nu + b + g}, \\ y^{-1} &= \frac{c\mu - b\nu + a + f}{a\mu - b\lambda + c + h}, \\ z^{-1} &= \frac{a\nu - c\lambda + b + g}{b\nu - c\mu + a + f}, \end{aligned}\right\}$$

together with the three other pairs obtained by changing the sign of one of the quantities f, g, h, and leaving the other two unaltered.

For example, let $\lambda = 0$, $\mu = 0$, and $\nu = 0$; then, taking $f = a$, $g = b$, and $h = c$, we have

$$x = \frac{c}{b}, \quad y = \frac{a}{c}, \quad z = \frac{b}{a},$$

and

$$x = \frac{b}{c}, \quad y = \frac{c}{a}, \quad z = \frac{a}{b};$$

or, in other words, the Brocard points are a pair of G points of the circumcircle ABC. Similarly, by taking $f = -a$, $g = b$, and $h = c$, we have

$$x = \frac{c}{b}, \quad y = 0, \quad z = \infty;$$

$$x = \frac{b}{c}, \quad y = \infty, \quad z = 0;$$

i.e., C, B are a pair of G points.

The formulæ are much simplified when the given circle is a pedal circle, say the pedal of (x, y, z), with regard to the triangle of reference. In this case, the isogonal coordinates of the G points are

$$\begin{array}{ccc} x, & y, & z, \\ x, & \dfrac{z + x\cos B}{z\cos B + x}, & \dfrac{x\cos C + y}{x + y\cos C}, \\ \dfrac{y\cos A + z}{y + z\cos A}, & y, & \dfrac{x + y\cos C}{x\cos C + y}, \\ \dfrac{y + z\cos A}{y\cos A + z}, & \dfrac{z\cos B + x}{z + x\cos B}, & z, \end{array}$$

together with the four inverse points obtained by writing $\frac{1}{x}$, $\frac{1}{y}$, $\frac{1}{z}$ for x, y, z in the above.

For example, let the given circle be the nine-point circle of the triangle of reference. This is known to be the pedal of the point $(2\cos A,\ 2\cos B,\ 2\cos C)$; so that we have for the other G points $\left(2\cos A,\ \frac{a}{c},\ \frac{a}{b}\right)$, $\left(\frac{b}{c},\ 2\cos B,\ \frac{b}{a}\right)$, $\left(\frac{c}{b},\ \frac{c}{a},\ 2\cos C\right)$, together with the inverse points $(\frac{1}{2}\sec A,\ \frac{1}{2}\sec B,\ \frac{1}{2}\sec C)$, $\left(\frac{1}{2}\sec A,\ \frac{c}{a},\ \frac{b}{a}\right)$, &c.

It is easily seen that these two sets of four points are respectively concyclic, the first set being on the Brocard circle

$$\Sigma\, bcx = a^2+b^2+c^2,$$

and the second on the orthocentroidal circle

$$\Sigma\, x\cos A = \tfrac{3}{2}.$$

This result can be generalized as follows, viz.,—Let O, P denote the centre of the circumcircle and orthocentre, respectively, of the triangle of reference ABC; H the point on the line PO produced, so that $OH=PO$. Then, if (x, y, z) be a point such that the line joining it to its inverse $\left(\frac{1}{x},\ \frac{1}{y},\ \frac{1}{z}\right)$ passes through H, the G points of the pedal circle of (x, y, z), with regard to the triangle ABC, can be divided into two sets of four concyclic points, which lie, respectively, on the circles

$$bc\frac{y+z\cos A}{x\cos C+y}(\xi-x)+ca\frac{z\cos B+x}{x+y\cos C}(\eta-y)+ab(\zeta-z)=0,$$

$$bc\frac{y^{-1}+z^{-1}\cos A}{x^{-1}\cos C+y^{-1}}(\xi-x^{-1})+ca\frac{z^{-1}\cos B+x^{-1}}{x^{-1}+y^{-1}\cos C}(\eta-y^{-1})+ab(\zeta-z^{-1})=0;$$

where ξ, η, ζ denote the current isogonal coordinates of a point, or

$$\Sigma\, a(\xi-\eta\zeta)=0.$$

For example, the point H satisfies the required condition, viz., that the line joining it to its inverse passes through H, whose coordinates are

$$x=\tfrac{1}{2}\frac{\cos A-\cos B\cos C}{\cos A\cos B\cos C},\quad y=\tfrac{1}{2}\frac{\cos B-\cos C\cos A}{\cos A\cos B\cos C},$$

$$z=\tfrac{1}{2}\frac{\cos C-\cos A\cos B}{\cos A\cos B\cos C}.$$

The G points of the pedal of H are H and the points

$$\left(\tfrac{1}{2}\frac{\cos A-\cos B\cos C}{\cos A\cos B\cos C},\ \frac{\cos C}{\cos A},\ \frac{\cos B}{\cos A}\right),$$

$$\left(\frac{\cos C}{\cos B},\ \tfrac{1}{2}\frac{\cos B-\cos C\cos A}{\cos A\cos B\cos C},\ \frac{\cos A}{\cos B}\right),\ \&c.$$

Four of these G points lie on the circumference of the circle

$$\Sigma\xi\sin A=\tfrac{1}{2}\tan A\tan B\tan C;$$

i.e., the circle described upon the segment PH as diameter.

The four inverse G points of the pedal in question lie on the circle

$$\Sigma\xi\tan A(\cos A-\cos B\cos C)=2\sin A\sin B\sin C,$$

i.e., the circle described upon OH' as diameter, where O and H' are the isogonals of P and H.

The above results lead to various theorems, among which the following may be noticed, viz.:—

(1) Let $A'B'C'$ be a triangle which has the same circumcircle and Brocard ellipse as a fixed triangle ABC; then the locus of one set of four G points of the nine-point circle of $A'B'C'$ is a fixed circle, viz., the Brocard circle of the triangle ABC.*

(2) Let $A'B'C'$ be a triangle which has the same circumcircle and self-polar circle as a fixed triangle ABC; then the locus of one set of four G points of the pedal of the fixed point H, with regard to the triangle $A'BC'$, is a fixed circle, viz., the circle described upon the segment PH as diameter.

On Clifford's paper "On Syzygetic Relations among the Powers of Linear Quantics." By Prof. CAYLEY. Received and Read November 12th, 1891.

The paper in question, originally printed, *Proc. Lond. Math. Soc.*, t. III. (1869), pp. 9–12, is reproduced No. XIV., pp. 119–122, of the *Mathematical Papers* (8vo, Lond., 1882), where it is immediately followed by the paper No. XV., "On Syzygetic Relations connecting

* With regard to this theorem, it should be observed that the Brocard circle of a triangle ABC is the locus of a point G whose pedal triangle has the same Brocard angle as ABC. [*April*, 1892.]

the Powers of Linear Quantics," pp. 123–129. The author, after referring to theorems in M. Paul Serret's *Géométrie de Direction* (Paris, 1869), proceeds as follows: —"By the use of Professor Sylvester's method of contravariant differentiation, I have arrived at certain extensions of these theorems, which I now proceed to explain," and he then states his Theorem I.: In order that a system of N points in a plane should all lie on a curve of the order n, it is sufficient that the p^{th} powers of their distances from an arbitrary line should satisfy a linear homogeneous relation, the number N being given by the formula

$$N = \tfrac{1}{2}\alpha n\,(n+3) + \tfrac{1}{2}\,(\beta+1)(\beta+2),$$

where α is the quotient, and β the remainder of the division of p by n, so that $p = \alpha n+\beta$, and $\beta < n$. And he then gives Theorem II., a like theorem as regards points in space; and, further, two Tables, A and B, for the values of N corresponding to given values of n and p in the two cases respectively.

Theorem I. is incorrect for the first value of N in Table A, viz., if $n = 1$, $p = 2$, then $N = 5$; the theorem here is: In order that a system of five points in a plane may lie in a line, the sufficient condition is that the squares of their distances from an arbitrary line shall satisfy a linear homogeneous relation; or, what is the same thing, if the squares of the distances of the five points satisfy a linear homogeneous relation, then the five points will lie in a line. The right conclusion is that four of the five points will lie in a line.

Before proceeding further, I slightly modify the form of the enunciation by defining (in the case of the plane figure) the power of a point to be its distance from an arbitrary line; or, what is the same thing, to be the nilfactum of the line-equation of the point; that is, for the point (x_1, y_1, z_1) the power is $\alpha x_1+\beta y_1+\gamma z_1$, where α, β, γ are arbitrary coefficients, or, if we please, line-coordinates. I say that $\alpha x_1+\beta y_1+\gamma z_1$ is the first power, $(\alpha x_1+\beta y_1+\gamma z_1)^2$ the second power, and so on. Clifford's Theorem I. thus is: If the p^{th} powers of the N points satisfy a homogeneous linear relation, the N points are on a curve of the n^{th} order.

In the case $n = 1$, $p = 2$, we have five points whose second powers satisfy a homogeneous linear relation; that is, if $(x_1, y_1, z_1), \ldots (x_5, y_5, z_5)$ are the coordinates of the five points respectively, we have

$$\lambda_1\,(\alpha x_1+\beta y_1+\gamma z_1)^2 \ldots +\lambda_5\,(\alpha x_5+\beta y_5+\gamma y_5)^2 = 0.$$

It is implicitly assumed that the points are distinct points, and we

thus exclude such solutions as

$$\lambda_1 = \lambda_2 = \lambda_3 = 0, \quad (x_4, y_4, z_4) = k\,(x_5, y_5, z_5), \quad k^3\lambda_4 + \lambda_5 = 0.$$

Here α, β, γ are arbitrary, and the relation is equivalent to the six equations

$$\begin{aligned} \lambda_1 x_1^2 \ldots + \lambda_5 x_5^2 &= 0, \\ y_1^2 &= 0, \\ z_1^2 &= 0, \\ y_1 z_1 &= 0, \\ z_1 x_1 &= 0, \\ x_1 y_1 &= 0; \end{aligned}$$

and hence, eliminating the λ's, we have the relation

$$\left\| \begin{matrix} x_1^2, & y_1^2, & z_1^2, & y_1 z_1, & z_1 x_1, & x_1 y_1 \\ x_2^2 & & & & & \\ x_3^2 & & & & & \\ x_4^2 & & & & & \\ x_5^2 & & & & & \end{matrix} \right\| = 0;$$

viz., this means that each of the determinants, obtained by selecting in any manner five out of the six columns, is $= 0$. This is a twofold relation, and thus it cannot be equivalent to the threefold relation

$$\left\| \begin{matrix} x_1, & x_2, & x_3, & x_4, & x_5 \\ y_1 & & & & \\ z_1 & & & & \end{matrix} \right\| = 0,$$

which expresses that the five points are in a line.

But we see further that the twofold relation expresses that the five points are such that we can, through them and an arbitrary sixth point (x_6, y_6, z_6), draw a determinate conic; and this is the case only if four of the five points are in a line; viz., the conic is then the line-pair composed of this line and of the line joining the remaining two points. The right conclusion therefore is that, if the above linear relation is satisfied, then four of the five points lie in a line.

Clifford's proof is rather indicated than carried out, but, from the

reference to contravariant differentiation, and from the second paper mentioned above, it must have been as follows:

Starting from the linear relation considered as an identity in (α, β, γ), and operating upon it with $p\partial_\alpha + q\partial_\beta + r\partial_\gamma$, p, q, r arbitrary coefficients, we obtain

$$\lambda_1 (px_1+qy_1+rz_1)(\alpha x_1+\beta y_1+\gamma z_1) \ldots$$
$$+\lambda_5 (px_5+qy_5+rz_5)(\alpha x_5+\beta y_5+\gamma z_5) = 0;$$

hence, determining the ratios of p, q, r, say by the equations

$$px_4+qy_4+rz_4 = 0, \quad px_5+qy_5+rz_5 = 0,$$

and, instead of $\lambda_1 (px_1+qy_1+rz_1)$, &c., writing Λ_1, &c., we have

$$\Lambda_1 (\alpha x_1+\beta y_1+\gamma z_1)+\Lambda_2 (\alpha x_2+\beta y_2+\gamma z_2)+\Lambda_3 (\alpha x_3+\beta y_3+\gamma z_3) = 0,$$

a homogeneous linear relation between the first powers of the three points (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), and we thus see that these points, say the points 1, 2, 3, are in a line; and, similarly, by different determinations of p, q, r, that the points 1, 2, 4 are in a line; and that the points 1, 2, 5 are in a line; that is, the points 3, 4 and 5 are each of them in the line 12 joining the points 1 and 2; that is, the points 1, 2, 3, 4, 5 are in a line.

But, if we examine the reasoning more closely, it appears that the first conclusion, 1, 2, 3 are in a line, depends on the assumption that neither 1, 2, nor 3 is in the line 45. Suppose, for instance, that 3 was in the line 45, the equations $px_4+qy_4+rz_4 = 0$, $px_5+qy_5+rz_5 = 0$, imply $px_3+qy_3+rz_3 = 0$, and we have

$$\lambda_1 (px_1+qy_1+rz_1)(\alpha x_1+\beta y_1+\gamma z_1)$$
$$+\lambda_2 (px_2+qy_2+rz_2)(\alpha x_2+\beta y_2+\gamma z_2) = 0,$$

or say $\quad \Lambda_1 (\alpha x_1+\beta y_1+\gamma z_1)+\Lambda_2 (\alpha x_2+\beta y_2+\gamma z_2) = 0,$

satisfied by $\quad (x_1, y_1, z_1) = k (x_2, y_2, z_2), \quad k\Lambda_1+\Lambda_2 = 0;$

that is, by making the points 1 and 2 coincident. Thus, if 3 be on the line 45 (and similarly if 1 or if 2 be on the line 45), we cannot infer that 1, 2, 3 are in a line.

I assume, therefore, that neither 1, 2, nor 3 is in the line 45; we here conclude, as above, that 1, 2, 3 are in a line. Suppose for a moment that 5 is not on this line; 4 cannot be on each of the lines 15, 25, 35, and I assume, in the first instance, that it is not on any

one of these lines; thus the points 1, 2, 4 are no one of them on the line 35, and hence, by the like reasoning, 1, 2, 4 are in a line; that is, 4 is on the line 12, or, the points 1, 2, 3, 4 are in a line. If, however, 4 is on one of the lines 15, 25, 35, say it is on 35; then it is not on 25, and no one of the points 1, 3, 4 is on 25; hence, by the like reasoning, 1, 2, 4 are in a line. In this case, however, 4 being, by supposition, on the line 35, can only be the point 3; that is, 3 and 4 would be coincident, a case which need not be considered. The correct conclusion from the reasoning, thus, is, not that 1, 2, 3, 4, 5 are in a line, but that some four of these five points, say 1, 2, 3, 4, are in a line.

It is easy to see that, if we have on a line four points, then there exists between the second powers of these points a linear homogeneous relation. For let the distances of the points from a fixed point of the line be a, b, c, d respectively; and let the arbitrary line meet the line in a point O at a distance r from the fixed point. The distances of the four points from the point O are thus equal to $r+a$, $r+b$, $r+c$, $r+d$ respectively; and hence, writing k for the squared cosine of the inclination of the two lines, the second powers of the four points are $= k(r+a)^2$, $k(r+b)^2$, $k(r+c)^2$, $k(r+d)^2$ respectively; and we have thus, between the second powers p_1, p_2, p_3, p_4 of the four points, tho homogeneous linear relation

$$\begin{vmatrix} p_1, & 1, & a, & a^2 \\ p_2, & 1, & b, & b^2 \\ p_3, & 1, & c, & c^2 \\ p_4, & 1, & d, & d^2 \end{vmatrix} = 0.$$

Moreover we can, in an infinite number of ways, form a linear combination $A_1p_1+A_2p_2+A_3p_3+A_4p_4$ of these second powers, which shall be independent of r, and have any given value whatever. We can therefore make this sum to be equal to the second power of any point 5 whatever (not on the line containing the four points) in relation to the arbitrary line; that is, given the four points in a line, and any other fifth point, we can establish between the second powers of these five points a linear homogeneous relation

$$A_1p_1+A_2p_2+A_3p_3+A_4p_4+A_5p_5=0,$$

with non-evanescent values for each of the coefficients A. We thus see how, to the linear homogeneous relation between the second

powers of the five points, there corresponds properly the non-symmetric relation: four of the five points are in a line.

The most simple cases are when $p = n$, viz., $p = n = 2$, $N = 6$; $p = n = 3$, $N = 10$, and generally $N = \frac{1}{2}(n+1)(n+2)$; and for these Clifford's theorem is easily verified. I have not examined the other cases, but it is probable that in each of them a correction is required.

December 10th, 1891.

Professor GREENHILL, F.R.S., President, in the Chair.

The auditor, Mr. Heppel, made his report, and a vote of thanks was unanimously accorded to him for the trouble he had taken in verifying the several items of the Treasurer's report.

Miss P. G. Fawcett, Newnham College, Cambridge; the Rev. C. H. W. Johns, M.A., late Scholar of Queens' College, Cambridge, St. Peter's College, Peterborough; Mr. A. E. Jolliffe, B.A., Fellow of Corpus Christi College, Oxford; Mr. A. Larmor, M.A., Fellow of Clare College, Cambridge; and Miss L. E. Worsfold, M.A., Associate of Bedford College, were elected Members.

The President paid a feeling tribute to the memory of his former colleague at the Royal Indian Engineering College, Cooper's Hill, the late Professor Wolstenholme, and gave a brief sketch of his mathematical work.

The following communications were made:—

The Equations of Propagation of Disturbances in Gyrostatically-Loaded Media: Dr. J. Larmor.

Theory of Elastic Wires: Mr. Basset.

Researches in the Calculus of Variations.—II. Discrimination of Maxima and Minima Solutions when the Variables are connected by Algebraical Equations, the limits being supposed fixed: Mr. E. P. Culverwell.

Note on the Algebraic Theory of Elliptic Transformation: Mr. J. Griffiths.

The Treasurer and Mr. J. Hammond made short impromptu communications. Messrs. Larmor, Basset, Kempe, Forsyth, Love,

S. Roberts, and the President, took part in the discussions on the papers.

The following presents were received:—

"Educational Times" for December.

"Journal of the Institute of Actuaries," No. CLXIII., October, 1891.

"The Educational Review," Vol. I., No. 1.

"Transactions of the Canadian Institute," Vol. II., Part 1, October, 1891.

"Bulletin des Sciences Mathématiques," November, 1891.

"Beiblätter zu den Annalen der Physik und Chemie," Band XV., Stück 10.

"Journal de l'École Polytechnique," Cahier 60.

"Reale Istituto Lombardo di Scienze e Lettere—Rendiconti," Vol. XXIII.

"Atti della Reale Accademia dei Lincei—Rendiconti," Vol. VII., Fasc. 8.

"Memorias y Revista de la Sociedad Cientifica 'Antonio Alzate,'" Tomo IV., Nos. 11 and 12.

"Memoires de la Société des Sciences Physiques et Naturelles de Bordeaux," Tome v., Cahier 2; Paris, 1890.

"Bollettino delle Pubblicazioni Italiane, ricevute per Diritto di Stampa," Nos. 141 and 142.

"Die Vergrösserung des Erdschattens bei Mondfinsternissen," von J. Hartmann; Leipzig, 1891.

"Ueber die Reihenentwickelungen der Potential-theorie," von Maxime Rôcher; Gottingen, 1891.

"Die Clausius'schen Coordinaten," von L. Kronecker.

"Observations Pluviométriques et Thermométriques faites dans le Département de la Gironde," Note de M. G. Rayet; Bordeaux, 1890.

On the Theory of Elastic Wires. *By* A. B. BASSET, M.A., F.R.S.

Received December 2nd, 1891. Read December 10th, 1891.

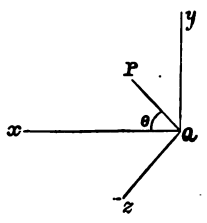
1. When a thin wire, whose natural form is any curve, is slightly bent and twisted, and the extension of the axis is neglected, the solution can be effected by means of a method originally due to Dr. Besant.* The stresses which act across any transverse section of the wire are six in number, and consist of (i.) a tension T along the tangent; (ii.) a shearing stress N_1 along the principal normal; (iii.) a shearing stress N_2 along the binormal; (iv.) a torsional couple

* *Quart. Journ. of Math.*, Vol. IV., p. 12.

H about the tangent; (v.) a couple G_1 about the principal normal; (vi.) a flexural couple G_2 about the binormal. By resolving the forces which act upon an element of the wire parallel to these directions, and taking moments about them, we obtain six equations connecting the six stresses, which are sufficient to determine the latter, when the form of the wire in its unstrained state is given. These equations do not, however, enable us to determine the form of the wire in its strained condition. In order to do this, we require certain additional equations, which will enable us to obtain four equations connecting the displacements of any point on the axis, together with a quantity β, such that $d\beta/ds$ measures the twist of the wire per unit of length. When the wire is inextensible, these equations, as we shall proceed to show, can be obtained by calculating the values of the three couples.

We shall confine our attention to *thin* wires, the natural forms of whose cross-sections are uniform and circular. The term *thin* must be understood as meaning, that the radius of the cross-section of the wire is small in comparison with the radius of curvature of its axis. The only perfectly rigorous method of dealing with a problem of this character, is to start with the general equations of motion of an elastic solid, expressed in terms of some convenient system of coordinates; to solve them; and then to determine the arbitrary constants which appear in the solution, by means of the boundary conditions. But the mathematical difficulties in the way of obtaining a solution by a method of this kind are of so serious a character, as to render it almost hopeless to attempt to attack the problem in this way. Under these circumstances, the only course open to us is to employ an indirect method, depending upon certain assumptions, which will now be explained.

2. Let Q be any point on the axis of the wire, when undeformed. Through Q, draw three axes Qx, Qy, Qz, which respectively coincide with the principal normal, binormal, and tangent to the axis at Q. Let u, v, w be the displacements of Q parallel to these directions. Our object is to express all the stresses in terms of u, v, w and their differential coefficients with regard to the arc s; but it will be found, as we proceed, that it will be necessary to introduce the quantity β.



Let P be any point on the cross-section through Q; let $QP = r$,

$PQx = \theta$, so that θ is the angle which QP makes with the principal normal at Q. Let u', v', w' be the displacements of P, along and perpendicular to QP, and parallel to the tangent at Q. Then u', v', w' are functions of the position of Q, and also of r and θ. Now it will subsequently appear that the portion V_0 of v', which does not involve θ, is zero when P coincides with Q, but that V_0/r is finite; the quantity β is the value of V_0/r when $r = 0$, and is evidently a function of the position of Q, such that $d\beta/ds$ measures the twist per unit of length.

Since u', v', w' are periodic functions of θ, we may evidently express them in the form of series of sines and cosines of multiples of θ, and may write

$$u' = U_0 + U_1 \cos\theta + U_1' \sin\theta + \dots \quad \dots\dots\dots\dots\dots(1),$$

where U_0, U_1 ... are functions of r, and the coordinates of Q. Since none of the quantities can be infinite when $r = 0$, they may be expanded in series of powers of r, and we may write

$$U_0 = u_0 + r\frac{du_0}{dr} + \tfrac{1}{2}r^2\frac{d^2u_0}{dr^2} + \dots \quad \dots\dots\dots\dots\dots(2),$$

where u_0, du_0/dr, &c. denote the values of U_0, dU_0/dr, &c., when $r = 0$.

The values of these quantities cannot be obtained by a direct method; I shall therefore adopt a procedure analogous to that employed in my papers on thin elastic shells.*

3. Let R be the traction, perpendicular to the middle surface, at any point of the substance of a thin elastic shell; and let S, T be the two shearing stresses, parallel to the middle surface, which tend to produce rotation about the lines of curvature of the latter. The boundary conditions require that R, S, and T should be zero at every point of the free surfaces of the shell, *provided the latter are not subjected to any surface pressures or tangential stresses.* At any point of the substance of the shell, R, S and T are not zero, except in special cases; but I have shown, in my papers on this subject, that if we are content with an approximate solution, which does not involve higher powers of the thickness than the cube, in the expressions for the kinetic and potential energies, R, S and T are small quantities, which may be *treated as zero*, inasmuch as their retention would lead to higher powers of the thickness than the cube. In the

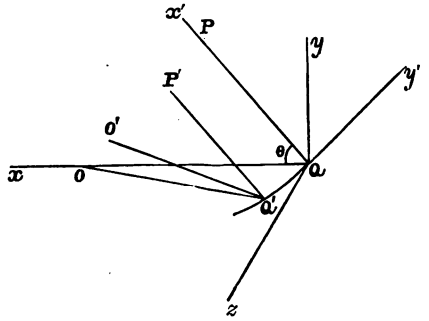
* *Proc. Lond. Math. Soc.*, Vol. xxi., p. 33; *Phil. Trans.*, 1890, p. 433.

case of a bent and twisted wire of small cross-section, let P, Q, R, S, T, U be the stresses at any point P of the substance of the wire, referred to QP, an axis perpendicular to QP in the plane of the section, and to the tangent to the central axis at Q. The boundary conditions require that P, T and U should vanish at every point of the surface of the wire, *provided the latter is not subjected to any surface pressures or tangential stresses.* In the case of a naturally straight wire, which is bent into a plane curve without extension, the potential energy and the flexural couple are proportional to the *fourth* power of the radius of the cross-section; I shall therefore assume, from analogy to the theory of thin shells, *that the three stresses P, T, U are small quantities, which may be treated as zero, provided the surface of the wire is not subjected to any surface forces, such as pressures or tangential stresses; and provided also, that the approximate expressions for the energy and the couples, do not include any higher powers of the radius of the cross-section than the fourth.*

To prevent misconception, it may be well to point out that this hypothesis does not assume that P, T and U are *actually zero.* The subsequent methods of procedure are very similar to those adopted in my papers on thin shells, and it will be found that several results drop out, in the course of the work, which can be verified by other methods.

4. We shall first obtain the equations of equilibrium in terms of the sectional stresses T, N_1, N_2, H, G_1, G_2.

Let Q' be a point on the axis near Q; let O, O' be the centres of principal curvature at Q, Q'; let $\delta\phi$, $\delta\eta$ be the angles of contingence and torsion at Q, so that $QOQ' = \delta\phi$, $OQ'O' = \delta\eta$; let ρ, σ be the radii of absolute curvature and torsion at Q; also let $T+\delta T$, $N_1+\delta N_1$,



&c. be the values of the sectional stresses at Q'. Let X, Y, Z be the components of the bodily forces per unit of length of the wire.

Resolving parallel to Qz, Qx, Qy, and taking moments about these lines, we obtain

$$\left.\begin{aligned} \frac{dT}{ds} - \frac{N_1}{\rho} + Z &= 0, \\ \frac{dN_1}{ds} - \frac{N_2}{\sigma} + \frac{T}{\rho} + X &= 0, \\ \frac{dN_2}{ds} + \frac{N_1}{\sigma} + Y &= 0, \\ \frac{dH}{ds} - \frac{G_1}{\rho} &= 0, \\ \frac{dG_1}{ds} - \frac{G_2}{\sigma} + \frac{H}{\rho} - N_2 &= 0, \\ \frac{dG_2}{ds} + \frac{G_1}{\sigma} + N_1 &= 0, \end{aligned}\right\} \quad \text{.....(3).}$$

5. We must now find expressions for the strains, in terms of the displacements.

Let u', v', w' be the displacements of P along QP, perpendicular to QP, in the direction in which θ increases, and parallel to the tangent at Q. Since θ is measured from the principal normal Qx at Q, it follows that, if $r+\delta r$, $\theta+\delta\theta$, $\rho\delta\varphi$ be the coordinates of a point P' in the cross-section through Q', then

$$O'Q'P' = \theta+\delta\theta, \quad \text{and} \quad OQ'P = \theta+\delta\theta+\delta\eta.$$

Let $u'+\delta u'$, $v'+\delta v'$, $w'+\delta w'$ be the displacements of P' along $Q'P'$, perpendicular to $Q'P'$, and perpendicular to the plane of the cross-section through Q'; also let $u'+\Delta u'$, $v'+\Delta v'$, $w'+\Delta w'$ be the displacements of P', parallel to the directions of u', v', w', that is, to Qx', Qy', Qz.

Then $$\delta x' = \delta r, \quad \delta y' = r\delta\theta, \quad \delta z = (\rho - r\cos\theta)\,\delta\phi \text{ .....(4);}$$

whence, in the limit,

$$\frac{\Delta u'}{\delta x'} = \frac{du'}{dr}, \quad \frac{\Delta v'}{\delta r'} = \frac{dv'}{dr}, \quad \frac{\Delta w'}{\delta x'} = \frac{dw'}{dr} \text{ ..... (5).}$$

Now $u'+\Delta u'/dy'.\,\delta y'$ is the displacement, parallel to Qx', of the point whose coordinates are r, $r\delta\theta$, 0; whence

$$\Delta u' = (u'+du'/d\theta\,.\,\delta\theta)\cos\delta\theta - (v'+dv'/d\theta\,.\,\delta\theta)\sin\delta\theta - u';$$

accordingly,

$$\frac{\Delta u'}{dy'} = \frac{1}{r}\frac{du'}{d\theta} - \frac{v'}{r} \text{ .....(6).}$$

Similarly,
$$\frac{\Delta v'}{dy'} = \frac{1}{r}\frac{dv'}{d\theta} + \frac{u'}{r} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7),$$
$$\frac{\Delta w'}{dy'} = \frac{1}{r}\frac{dw'}{d\theta} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8).$$

Let P' be a point in the plane $OQ'P'$, such that
$$Q'P' = QP, \quad \text{and} \quad P'Q'O = PQO.$$
Then the coordinates of P' are r, $\theta - \delta\eta$, and $\rho\delta\phi$; whence the value of $\delta u'$ at P' is
$$-\frac{du'}{d\theta}\delta\eta + \frac{du'}{d\varphi}\delta\phi,$$
and similarly for $\delta v'$, $\delta w'$.

Now $u' + \Delta u'/dz \,.\, \delta z$ is the component displacement of P', parallel to QP; whence
$$\Delta u' = \{(u'+\delta u')\cos\theta - (v'+\delta v')\sin\theta\}\cos\delta\phi\cos\theta$$
$$+ \{(u'+\delta u')\sin\theta + (v'+\delta v')\cos\theta\}\sin\theta + (w'+\delta w')\sin\delta\phi\cos\theta - u'$$
$$= \delta u' + w'\cos\theta\,\delta\phi;$$
accordingly,
$$\frac{\Delta u'}{dz} = \frac{1}{\rho - r\cos\theta}\left(\frac{du'}{d\phi} - \frac{\rho}{\sigma}\frac{du'}{d\theta} + w'\cos\theta\right) \quad \ldots\ldots\ldots (9).$$

Similarly,
$$\Delta v' = \delta v' - w'\sin\theta\,\delta\phi;$$
whence
$$\frac{\Delta v'}{dz} = \frac{1}{\rho - r\cos\theta}\left(\frac{dv'}{d\phi} - \frac{\rho}{\sigma}\frac{dv'}{d\theta} - w'\sin\theta\right) \ldots\ldots\ldots (10).$$

And
$$\Delta w' = \delta w' - (u'\cos\theta - v'\sin\theta)\,\delta\phi,$$
whence
$$\frac{\Delta w'}{dz} = \frac{1}{\rho - r\cos\theta}\left\{\frac{dw'}{d\phi} - \frac{\rho}{\sigma}\frac{dw'}{d\theta} - (u'\cos\theta - v'\sin\theta)\right\} \ldots (11).$$

We shall find it convenient to introduce, in certain portions of the work, two new quantities β and γ, such that
$$v' = r\beta, \quad w' = (\rho - r\cos\theta)\,\gamma \quad \ldots\ldots\ldots\ldots\ldots (12),$$
so that β is the angular displacement about the tangent to the axis of the wire, and γ is the angular displacement about a line through O parallel to the binormal. We thus obtain the following equations

for the strains, in terms of the displacements, viz. :—

$$e = \frac{du'}{dr} \quad \text{.....(13),}$$

$$f = \frac{1}{r}\frac{dv'}{d\theta} + \frac{u'}{r} \quad \text{.....(14),}$$

$$g = \frac{1}{\rho - r\cos\theta}\left(\frac{dw'}{d\phi} - u'\cos\theta + v'\sin\theta\right) \quad \text{.....(15),}$$

$$a = \frac{\rho - r\cos\theta}{r}\frac{d\gamma}{d\theta} + \frac{1}{\rho - r\cos\theta}\frac{dv'}{d\phi} - \frac{\rho}{\sigma(\rho - r\cos\theta)}\frac{dv'}{d\theta} \quad \text{...(16),}$$

$$b = \frac{1}{\rho - r\cos\theta}\frac{du'}{d\phi} + (\rho - r\cos\theta)\frac{d\gamma}{dr} - \frac{\rho}{\sigma(\rho - r\cos\theta)}\frac{du'}{d\theta} \quad \text{...(17),}$$

$$c = \frac{dv'}{dr} - \frac{v'}{r} + \frac{1}{r}\frac{du'}{d\theta} \quad \text{.....(18).}$$

6. The values of the three couples are evidently

$$H = n\int_0^c\int_0^{2\pi} ar^2\,dr\,d\theta,$$

$$G_1 = \int_0^c\int_0^{2\pi} Rr^2\sin\theta\,dr\,d\theta,$$

$$G_2 = -\int_0^c\int_0^{2\pi} Rr^2\cos\theta\,dr\,d\theta,$$

where, in these equations, c denotes the radius of the cross-section of the wire.

From these equations, it appears that the terms in a, which involve θ, contribute nothing to the value of H, since they disappear upon integration. Similarly, only those terms in R, which involve $\sin\theta$ and $\cos\theta$, contribute anything to the values of G_1, G_2. We may, therefore, in calculating the couples, omit all terms involving sines and cosines of multiples of θ higher than the first; and we may also omit powers of r higher than the first, in calculating the strains, since such terms would introduce into the values of the couples higher powers of c than the fourth.

7. Since all the quantities are periodic functions of θ, they may be expanded in series of sines and cosines of θ, the coefficients of which are functions of r and s. We may, therefore, write

$$\left.\begin{aligned} u' &= U_0 + U_1\cos\theta + U_1'\sin\theta + \ldots \\ v' &= V_0 + V_1\cos\theta + V_1'\sin\theta + \ldots \\ \gamma &= \Gamma_0 + \Gamma_1\cos\theta + \Gamma_1'\sin\theta + \ldots \end{aligned}\right\} \quad \text{.....(19).}$$

Let u, v, w denote the displacements of Q, parallel to the principal normal, the binormal, and the tangent. Then, when $r = 0$,

$$u = U_0 + U_1 = -U_0 + U_1 = V_0 - V_1' = -V_0 - V_1';$$

which requires that, when $r = 0$,

$$U_0 = 0, \quad V_0 = 0, \quad u = U_1 = -V_1' \ldots\ldots\ldots\ldots\ldots\ldots (20).$$

Similarly, when $r = 0$,

$$v = V_0 + V_1 = -V_0 + V_1 = U_0 + U_1' = -U_0 + U_1';$$

which requires that

$$V_0 = 0, \quad U_0 = 0, \quad v = V_1 = U_1' \ldots\ldots\ldots\ldots\ldots\ldots (21).$$

Since U_0, V_0 vanish when $r = 0$, it follows that these quantities must contain r as a factor; we may therefore write the first two of (19) in the form

$$\left.\begin{aligned} u' = r\left(u_0 + r\frac{du_0}{dr}\right) &+ \left(u + r\frac{du_1}{dr} + \tfrac{1}{2}r^2\frac{d^2u_1}{dr^2}\right)\cos\theta \\ &+ \left(v + r\frac{du_1'}{dr} + \tfrac{1}{2}r^2\frac{d^2u_1'}{dr^2}\right)\sin\theta, \\ v' = r\left(\beta + r\frac{d\beta}{dr}\right) &+ \left(v + r\frac{dv_1}{dr} + \tfrac{1}{2}r^2\frac{d^2v_1}{dr^2}\right)\cos\theta \\ &+ \left(-u + r\frac{dv_1'}{dr} + \tfrac{1}{2}r^2\frac{d^2v_1'}{dr^2}\right)\sin\theta, \end{aligned}\right\} \ldots\ldots\ldots (22).$$

It is also necessary that the strains should be finite when $r = 0$; and, on substituting the above values of u', v' in (13), (14) and (18), it will be found that this condition is satisfied. It also follows, from (16), that γ must be of the form

$$\gamma = \gamma_0 + r\frac{d\gamma_0}{dr} + \tfrac{1}{2}r^2\frac{d^2\gamma_0}{dr^2} + r\left(\gamma_1 + r\frac{d\gamma_1}{dr}\right)\cos\theta + r\left(\gamma_1' + r\frac{d\gamma_1'}{dr}\right)\sin\theta \ldots (23).$$

In (22) and (23), u_0, du_0/dr, &c., are the values of the quantities on the axis of the wire, and are therefore functions of s (or φ) alone.

The values of e and f are

$$e = u_0 + 2r\frac{du_0}{dr} + \left(\frac{du_1}{dr} + r\frac{d^2u_1}{dr^2}\right)\cos\theta + \left(\frac{du_1'}{dr} + r\frac{d^2u_1'}{dr^2}\right)\sin\theta \ldots\ldots (24),$$

$$f = u_0 + r\frac{du_0}{dr} + \left\{\frac{du_1}{dr} + \frac{dv_1'}{dr} + \tfrac{1}{2}r\left(\frac{d^2u_1}{dr^2} + \frac{d^2v_1'}{dr^2}\right)\right\}\cos\theta + \left\{\frac{du_1'}{dr} - \frac{dv_1}{dr} + \tfrac{1}{2}r\left(\frac{d^2u_1'}{dr^2} - \frac{d^2v_1}{dr^2}\right)\right\}\sin\theta \ldots\ldots (25).$$

Let σ_1, σ_2 be the extensions, at any point of the axis, along the normal (measured inwards), and the binormal. Then, when $r = 0$ and $\theta = 0$,

$$e = \sigma_1, \quad f = \sigma_2;$$

when $r = 0$ and $\theta = \frac{1}{2}\pi$,

$$e = \sigma_2;$$

when $r = 0$ and $\theta = \frac{3}{2}\pi$,

$$f = \sigma_1.$$

Accordingly, we find that

$$\left.\begin{aligned} \frac{du_1}{dr} &= \sigma_1 - u_0, \quad \frac{dv_1}{dr} = \sigma_1 + \sigma_2 - 2u_0, \\ \frac{du_1'}{dr} &= \sigma_2 - u_0, \quad \frac{dv_1'}{dr} = \sigma_2 - \sigma_1, \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (26).$$

8. We must now introduce the conditions that the stresses P, T, U are small quantities, which, to the above order of approximation may be treated as zero.

By (18), (22), and (26), the condition that $U = 0$ gives

$$r\frac{d\beta}{dr} + \left\{\sigma_2 - u_0 + \tfrac{1}{2}r\left(\frac{d^2v_1}{dr^2} + \frac{d^2u_1'}{dr^2}\right)\right\}\cos\theta - \left\{\sigma_1 - u_0 + \tfrac{1}{2}r\left(\frac{d^2u_1}{dr^2} - \frac{d^2v_1'}{dr^2}\right)\right\}\sin\theta = 0.$$

Since this equation must be true for all values of r and θ, we obtain

$$\left.\begin{aligned} \frac{d\beta}{dr} &= 0, \quad \sigma_1 = \sigma_2 = u_0, \\ \frac{d^2v_1}{dr^2} &= -\frac{d^2u_1'}{dr^2}, \quad \frac{d^2u_1}{dr^2} = \frac{d^2v_1'}{dr^2}, \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots (27).$$

Equations (22), (24) and (25) may now be written

$$\left.\begin{aligned} u' &= r\left(\sigma_1 + r\frac{du_0}{dr}\right) + \left(u + \tfrac{1}{2}r^2\frac{d^2u_1}{dr^2}\right)\cos\theta + \left(v - \tfrac{1}{2}r^2\frac{d^2v_1}{dr^2}\right)\sin\theta, \\ v' &= r\beta + \left(v + \tfrac{1}{2}r^2\frac{d^2v_1}{dr^2}\right)\cos\theta + \left(-u + \tfrac{1}{2}r^2\frac{d^2u_1}{dr^2}\right)\sin\theta, \end{aligned}\right\} \quad \ldots(28),$$

$$\left.\begin{aligned} e &= \sigma_1 + 2r\frac{du_0}{dr} + r\frac{d^2u_1}{dr^2}\cos\theta - r\frac{d^2v_1}{dr^2}\sin\theta, \\ f &= \sigma_1 + r\frac{du_0}{dr} + r\frac{d^2u_1}{dr^2}\cos\theta - r\frac{d^2v_1}{dr^2}\sin\theta, \end{aligned}\right\} \quad \ldots\ldots\ldots (29).$$

By (17), (23), and (28), the condition that $T = 0$ gives

$$\left(1+\frac{r}{\rho}\cos\theta\right)\left\{\frac{r}{\rho}\frac{d\sigma_1}{d\phi}+\left(\frac{1}{\rho}\frac{du}{d\phi}-\frac{v}{\sigma}\right)\cos\theta+\left(\frac{1}{\rho}\frac{dv}{d\phi}+\frac{u}{\sigma}\right)\sin\theta\right\}$$

$$+(\rho-r\cos\theta)\left\{\frac{d\gamma_0}{dr}+r\frac{d^2\gamma_0}{dr^2}+\left(\gamma_1+2r\frac{d\gamma_1}{dr}\right)\cos\theta+\left(\gamma_1'+2r\frac{d\gamma_1'}{dr}\right)\sin\theta\right\}$$

$$=0.$$

Since we neglect powers of r^2 in the stress equations, it follows, by equating the coefficients of r, $\cos\theta$, $\sin\theta$, to zero, that we obtain

$$\frac{d\gamma_0}{dr}=0,\quad \frac{d\gamma_1}{dr}=0,\quad \frac{d\gamma_1'}{dr}=0,$$

$$\frac{d^2\gamma_0}{dr^2}=-\frac{1}{\rho^2}\frac{d\sigma_1}{d\phi}-\frac{1}{\rho^2}\left(\frac{1}{\rho}\frac{du}{d\phi}-\frac{v}{\sigma}\right),$$

$$\gamma_1=-\frac{1}{\rho}\left(\frac{1}{\rho}\frac{du}{d\phi}-\frac{v}{\sigma}\right),$$

$$\gamma_1'=-\frac{1}{\rho}\left(\frac{1}{\rho}\frac{dv}{d\phi}+\frac{u}{\sigma}\right);$$

whence

$$\gamma=\gamma_0-\frac{r^2}{2\rho^2}\left(\frac{d\sigma_1}{d\phi}+\frac{1}{\rho}\frac{du}{d\phi}-\frac{v}{\sigma}\right)$$

$$-\frac{r}{\rho}\left(\frac{1}{\rho}\frac{du}{d\phi}-\frac{v}{\sigma}\right)\cos\theta-\frac{r}{\rho}\left(\frac{1}{\rho}\frac{dv}{d\phi}+\frac{u}{\sigma}\right)\sin\theta\ldots\ldots(30).$$

And therefore, by (12),

$$w'=w-\frac{r^2}{2\rho}\frac{d\sigma_1}{d\phi}-r\left(\frac{1}{\rho}\frac{du}{d\phi}+\frac{w}{\rho}-\frac{v}{\sigma}\right)\cos\theta-r\left(\frac{1}{\rho}\frac{dv}{d\phi}+\frac{u}{\sigma}\right)\sin\theta\ldots(31).$$

By (15), (28), and (31), the value of g is

$$g=\sigma_3-\frac{r}{\rho}\left\{\frac{d}{d\phi}\left(\frac{1}{\rho}\frac{du}{d\phi}+\frac{w}{\rho}-\frac{v}{\sigma}\right)+\sigma_1-\sigma_3\right\}\cos\theta$$

$$-\frac{r}{\rho}\left\{\frac{d}{d\phi}\left(\frac{1}{\rho}\frac{dv}{d\phi}+\frac{u}{\sigma}\right)-\beta\right\}\sin\theta\quad\ldots\ldots\ldots\ldots\quad(32),$$

where σ_3 is the extension of the axis along the tangent; so that we have

$$\sigma_3=\frac{1}{\rho}\frac{dw}{d\phi}-\frac{u}{\rho}\quad\ldots\ldots\ldots\ldots\ldots\ldots\quad(33).$$

Now

$$P=(m+n)\,e+(m-n)\,(f+g);$$

whence, if $P = 0$,

$$(m+n)\left\{\sigma_1 + 2r\frac{du_0}{dr} + r\frac{d^2u_1}{dr^2}\cos\theta - r\frac{d^2v_1}{dr^2}\sin\theta\right\}$$

$$+(m-n)\left\{\sigma_1 + r\frac{du_0}{dr} + r\frac{d^2u_1}{dr^2}\cos\theta - r\frac{d^2v_1}{dr^2}\sin\theta + \sigma_3\right.$$

$$-\frac{r}{\rho}\left[\frac{d}{d\phi}\left(\frac{1}{\rho}\frac{du}{d\phi} + \frac{w}{\rho} - \frac{v}{\sigma}\right) + \sigma_1 - \sigma_3\right]\cos\theta$$

$$\left.-\frac{r}{\rho}\left[\frac{d}{d\phi}\left(\frac{1}{\rho}\frac{dv}{d\phi} + \frac{u}{\sigma}\right) - \beta\right]\sin\theta\right\} = 0.$$

Accordingly,

$$\left.\begin{aligned} 2m\sigma_1 + (m-n)\sigma_3 &= 0, \\ \frac{du_0}{dr} &= 0, \\ \frac{2m}{m-n}\frac{d^2u_1}{dr^2} &= \frac{1}{\rho}\frac{d}{d\phi}\left(\frac{1}{\rho}\frac{du}{d\phi} + \frac{w}{\rho} - \frac{v}{\sigma}\right) + \frac{\sigma_1 - \sigma_3}{\rho}, \\ -\frac{2m}{m-n}\frac{d^2v_1}{dr^2} &= \frac{1}{\rho}\frac{d}{d\phi}\left(\frac{1}{\rho}\frac{dv}{d\phi} + \frac{u}{\sigma}\right) - \frac{\beta}{\rho}, \end{aligned}\right\} \quad \ldots (34).$$

From the first of these equations, we see that the ratio of lateral contraction to longitudinal extension is equal to $(m-n)/2m$, (Poisson's ratio), which is a well-known result.

We have now completely determined all the unknown quantities; whence, denoting the right-hand sides of the last two of (34) by λ and μ, and collecting our results, we obtain

$$\left.\begin{aligned} u' &= r\sigma_1 + u\cos\theta + v\sin\theta + \frac{(m-n)r^2}{4m}(\lambda\cos\theta + \mu\sin\theta), \\ v' &= r\beta + v\cos\theta - u\sin\theta + \frac{(m-n)r^2}{4m}(\lambda\sin\theta - \mu\cos\theta), \end{aligned}\right\} \ldots (35),$$

$$e = f = \sigma_1 + \frac{(m-n)r}{2m}(\lambda\cos\theta + \mu\sin\theta) \quad \ldots\ldots\ldots (36),$$

whilst the values of γ, w', and g are given by (30), (31), and (32).

9. We can now calculate the three couples.

By (16), (30), and (35),

$$a = \frac{r}{\rho}\left(\frac{d\beta}{d\phi} + \frac{1}{\rho}\frac{dv}{d\phi} + \frac{u}{\sigma}\right) \quad \ldots\ldots\ldots\ldots\ldots (37).$$

As we shall no longer require to use the value of c for the strain

given by (18), we shall henceforth employ this letter to denote the radius of the cross-section of the wire; whence

$$H = n\int_0^c\int_0^{2\pi} ar^3 dr\,d\theta$$

$$= \tfrac{1}{2}\pi nc^4\left(\frac{1}{\rho}\frac{d\beta}{d\phi} + \frac{1}{\rho^2}\frac{dv}{d\phi} + \frac{u}{\rho\sigma}\right) \quad \ldots\ldots\ldots\ldots\ldots (38).$$

Also,

$$\left.\begin{aligned} G_1 &= \int_0^c\int_0^{2\pi} Rr^3\sin\theta\,dr\,d\theta, \\ G_2 &= -\int_0^c\int_0^{2\pi} Rr^3\cos\theta\,dr\,d\theta, \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots (39).$$

Now $$R = (m+n)\,g + (m-n)\,(e+f);$$

whence, by (32), (34), and (36),

$$G_1 = -\frac{(3m-n)\,n\pi c^4}{4m}\left\{\frac{1}{\rho}\frac{d}{d\phi}\left(\frac{1}{\rho}\frac{dv}{d\phi} + \frac{u}{\sigma}\right) - \frac{\beta}{\rho}\right\} \quad \ldots\ldots\ldots\ldots\ldots (40),$$

$$G_2 = \frac{(3m-n)\,n\pi c^4}{4m}\left\{\frac{1}{\rho}\frac{d}{d\phi}\left(\frac{1}{\rho}\frac{du}{d\phi} + \frac{w}{\rho} - \frac{v}{\sigma}\right) + \frac{\sigma_1-\sigma_3}{\rho}\right\} \quad \ldots (41).$$

Equations (38), (40) and (41) give the values of the three couples in terms of the displacements and β.

10. We have now obtained all the materials we require for the complete solution of the problem, *provided the wire is inextensible.* For if we substitute the values of H and G_1 in the fourth of (3), we obtain one equation. In the second place, substitute the values of H, G_1, G_2 in the last two, then substitute the values of N_1, N_2, thus obtained, in the first three of (3), and eliminate T. This gives two more equations. We have therefore, three equations, which, together with the condition of inextensibility, give four equations to determine the four unknown quantities u, v, w, β.

If the wire were supposed to be extensible, it would be necessary to proceed to a higher degree of approximation, so as to include terms involving r^3 in the expressions for the strains; but the resulting equations would be of a complicated character, and would be of very little practical value, since, in most applications to spiral springs, we may suppose the spring to be inextensible.

11. The preceding method does not enable us to calculate the potential energy due to strain, even when the wire is inextensible. In this case, the values of the strains, given by (36) and (37), are

proportional to r, and the resulting value of W would be

$$W = \tfrac{1}{8}\pi c^4 \int \{2na^2 + q(\lambda^2+\mu^2)\}\, ds,$$

where q is Young's modulus. It might, however, happen that e, f, g, a contain terms involving sines and cosines of multiples of θ higher than the first, whose coefficients contain the *first* power of r; and, if such were the case, these terms would not disappear on integration. The correct expression for the potential energy, when there is extension as well as flexion and torsion, might probably be obtained by proceeding to a higher degree of approximation, but the work would be complicated. From analogy to the case of a thin plate, we should infer that the complete value of the potential energy consists of three distinct terms, viz.:—

(i.) A term

$$\tfrac{1}{2}\pi c^2 q\sigma_s^2 = \tfrac{1}{2}\pi c^2 q\left(\frac{1}{\rho}\frac{dw}{d\varphi} - \frac{u}{\rho}\right)^2$$

which is proportional to the square of the radius, and depends solely upon the extension of the axis, and not upon flexion or torsion.

(ii.) A term depending principally upon flexion and torsion, but into which extensional terms also enter. These two terms are analogous to the first two terms in the expression for the potential energy of a thin cylindrical shell.*

(iii.) A term involving quantities which depend upon extension, torsion, and flexion, and which disappears when the wire is inextensible. This term would correspond to the last two lines of the expression for the potential energy of a cylindrical shell; and its value could not be calculated without proceeding to a higher order of approximation.

12. We shall now calculate the change of curvature due to strain.† Let Q_2, Q, Q_1 be three adjacent points on a tortuous curve. Then, if ϕ and η are the angles of contingence and torsion at Q, the coordinates of Q_1, relatively to Q, referred to the normal, binormal, and tangent, at Q, are

$$\tfrac{1}{2}\rho\phi^2, \quad 0, \quad \rho\phi;$$

* *Phil. Trans.*, 1890. Equation (24), page 443.

† When a wire or a surface is slightly deformed in any manner, it is evident that a number of expressions might be obtained, for various geometrical quantities relating to the new curve or surface (such as the changes of curvature, torsion, and the like), in terms of the displacements u, v, w. The subject is one of great interest from a geometrical point of view, and has been considered in a paper by Clerk-Maxwell, "On the Transformation of Surfaces by Bending," *Trans. Camb. Phil. Soc.*, Vol. IX.; and *Scientific Papers*, Vol. I., p. 80.

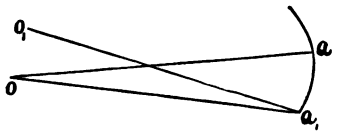
powers of ϕ higher than ϕ^2 being neglected. The coordinates of Q_2 are obtained by changing the sign of ϕ.

Let Q_2, Q, Q_1 be displaced to R_2, R, R_1, and let u, v, w be the displacements of Q, parallel to the normal, binormal, and tangent. Then, since we have to take into account infinitesimals of the second order, it follows that

$$QOQ_1 = \frac{s}{\rho} - \frac{s^2\rho'}{2\rho^2} = \phi,$$

and

$$OQ_1O_1 = \frac{s}{\sigma} - \frac{s^2\sigma'}{2\sigma^2} = \eta,$$



where the accents denote differentiation with respect to the arc s. Whence, if $u+\delta u$, $v+\delta v$, $w+\delta w$ be the displacements of Q_1, parallel to the normal, principal normal, and tangent, the coordinates of R_1, relatively to R, are

$$\{(u+\delta u)\cos\eta - (v+\delta v)\sin\eta\}\cos\phi + (w+\delta w)\sin\phi + \frac{s^2}{2\rho} - u = x.$$

Now

$$\delta u = su' + \tfrac{1}{2}s^2u'', \text{ \&c.},$$

whence, substituting the values of δu, η, ϕ, &c., and neglecting cubes of small quantities, we obtain

$$x = s\left(u' - \frac{v}{\sigma} + \frac{w}{\rho}\right) + \tfrac{1}{2}s^2\left(\frac{1}{\rho} + u'' - \frac{u}{\rho^2} - \frac{u}{\sigma^2} - \frac{2v'}{\sigma} + \frac{v\sigma'}{\sigma^2} + \frac{2w'}{\rho} - \frac{w\rho'}{\rho^2}\right).$$

By proceeding in a similar manner, we can show that the coordinates of R_1, relatively to R, are

$$As + A's^2, \quad Bs + B's^2, \quad Cs + C's^2,$$

where

$$A = u' - \frac{v}{\sigma} + \frac{w}{\rho},$$

$$B = v' + \frac{u}{\sigma},$$

$$C = 1 + w' - \frac{u}{\rho},$$

$$A' = \tfrac{1}{2}\left(\frac{1}{\rho} + u'' - \frac{u}{\rho^2} - \frac{u}{\sigma^2} - \frac{2v'}{\sigma} + \frac{v\sigma'}{\sigma^2} + \frac{2w'}{\rho} - \frac{w\rho'}{\rho^2}\right),$$

$$B' = \tfrac{1}{2}\left(v'' - \frac{v}{\sigma^2} + \frac{2u'}{\sigma} - \frac{u\sigma'}{\sigma^2}\right),$$

$$C' = \tfrac{1}{2}\left(w'' - \frac{w}{\rho} - \frac{2u'}{\rho} + \frac{2v}{\rho\sigma}\right).$$

The coordinates of R_2, relatively to R, will be obtained by changing the sign of s.

If $\delta\epsilon'$ be the angle of contingence of the deformed curve,

$$\cos R_1RR_2 = -\cos\delta\epsilon' = \frac{RR_1^2+RR_2^2-R_1R_2^2}{2RR_1.RR_2};$$

whence
$$\delta\epsilon'^2 = \frac{4(A'^2+B'^2+C'^2)s^2}{(A^2+B^2+C^2)\rho^2};$$

and
$$RR_1^2 = (A^2+B^2+C^2)s^2/\rho^2.$$

Substituting the values of A, A', &c., and reducing, we obtain

$$\frac{1}{\rho'} = \frac{2A'}{C};$$

or
$$\frac{1}{\rho'}-\frac{1}{\rho} = \frac{d^2u}{ds'}-\frac{u}{\sigma^2}-\frac{2v'}{\sigma}+\frac{v\sigma'}{\sigma^2}+\frac{w'}{\rho}-\frac{w\rho'}{\rho^2} \quad \ldots\ldots\ldots\ldots (42)$$

$$= \frac{1}{\rho}\frac{d}{d\phi}\left(\frac{1}{\rho}\frac{du}{d\phi}+\frac{w}{\rho}-\frac{v}{\sigma}\right)-\frac{u}{\sigma^2}-\frac{v'}{\sigma} \quad \ldots\ldots\ldots (43).$$

Comparing this result with (41), we see that when the wire is a plane curve, and is inextensible, the flexural couple is proportional to the change of curvature. The more rigorous theory, developed in the present paper, accordingly agrees with the imperfect theory given in the books on elasticity.*

13. We are now in a position to obtain the equations of motion.

Let h be the density, then, in the first three of (31), we must write

$$Z-h\pi c^2\ddot{w}, \quad X-h\pi c^2\ddot{u}, \quad Y-h\pi c^2\ddot{v},$$

for Z, X, and Y.

If the wire were subjected to impressed forces, which are reducible to a resultant couple as well as to a resultant force, it would be necessary to add terms of the form N, L, M to the left-hand sides of the last three of (3); whence, if the wire is in motion, we shall have

$$-N = h\iint\ddot{\beta}r^3\,dr\,d\theta = \tfrac{1}{2}\pi hc^4\ddot{\beta},$$

$$-L = h\iint\ddot{w}r^2\sin\theta\,dr\,d\theta = -\tfrac{1}{4}\pi hc^4\left(\frac{d\ddot{v}}{ds}+\frac{\ddot{u}}{\sigma}\right),$$

$$-M = -h\iint\ddot{w}r^2\cos\theta\,dr\,d\theta = \tfrac{1}{4}\pi hc^4\left(\frac{d\ddot{u}}{ds}+\frac{\ddot{w}}{\rho}-\frac{\ddot{v}}{\sigma}\right);$$

* I am indebted to Mr. Lachlan for examining the analysis of this section.

accordingly, the equations of motion are

$$\left.\begin{aligned}
\frac{dT}{ds}-\frac{N_1}{\rho} &= h\pi c^2\ddot{w}-Z,\\
\frac{dN_1}{ds}-\frac{N_2}{\sigma}+\frac{T}{\rho} &= h\pi c^2\ddot{u}-X,\\
\frac{dN_2}{ds}+\frac{N_1}{\sigma} &= h\pi c^2\ddot{v}-Y,\\
\frac{dH}{ds}-\frac{G_1}{\rho} &= \tfrac{1}{2}\pi hc^4\ddot{\beta},\\
\frac{dG_1}{ds}-\frac{G_2}{\sigma}+\frac{H}{\rho}-N_2 &= -\tfrac{1}{4}\pi hc^4\left(\frac{d\ddot{v}}{ds}+\frac{\ddot{u}}{\sigma}\right),\\
\frac{dG_2}{ds}+\frac{G_1}{\sigma}+N_1 &= \tfrac{1}{4}\pi hc^4\left(\frac{d\ddot{u}}{ds}+\frac{\ddot{w}}{\rho}-\frac{\ddot{v}}{\sigma}\right),
\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (44).$$

14. We shall now apply (44) to investigate the vibrations of a complete circular wire.

In the first place, the wire is capable of radial vibrations, which depend solely upon extension. Here $w = v = 0$, and

$$W = \tfrac{1}{2}\pi c^2 q u^2/a^2,$$

where a and c are the radii of the axis and cross-section respectively. Whence the equation of motion

$$\rho\frac{d^2u}{dt^2}+qu/a^2 = 0,$$

where ρ now denotes the density; whence

$$\tau = 2\pi a\,(\rho/q)^{\frac{1}{2}}.$$

The notes yielded by these vibrations are of high pitch, and of not much physical interest.

15. To investigate the remaining types of vibration, we must employ the general equations of motion.

If M be the mass of the wire per unit of length, equations (44)

become

$$\left.\begin{aligned}
\frac{dT}{d\phi} - N_1 &= Ma\ddot{w},\\
\frac{dN_1}{d\phi} + T &= Ma\ddot{u},\\
\frac{dN_2}{d\phi} &= Ma\ddot{v},\\
\frac{dH}{d\phi} - G_1 &= \tfrac{1}{2}Mc^2 a\ddot{\beta},\\
\frac{dG_1}{d\phi} + H - N_2 a &= -\tfrac{1}{4}Mc^2 \frac{d\ddot{v}}{d\phi},\\
\frac{dG_2}{d\phi} + N_1 a &= \tfrac{1}{4}Mc^2\left(\frac{d\ddot{u}}{d\phi} + \ddot{w}\right),
\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (45);$$

also

$$\frac{dw}{d\phi} = u \quad \ldots\ldots\ldots\ldots (46).$$

The values of the couples may be written

$$H = \frac{Mnc^2}{2\rho a}\left(\frac{d\beta}{d\phi} + \frac{1}{a}\frac{dv}{d\phi}\right),$$

$$G_1 = -\frac{Mqc^2}{4\rho a}\left(\frac{1}{a}\frac{d^2v}{d\phi^2} - \beta\right),$$

$$G_2 = \frac{Mqc^2}{4\rho a^2}\left(\frac{d^2u}{d\phi^2} + u\right).$$

If we substitute the value of G_2 in the sixth of (45), and then eliminate N_1, T, and u, from the first and second (45), the resulting equation will involve u alone, and not v or β. This equation determines the flexural vibrations discussed by Hoppe and Lord Rayleigh, whose frequency is determined by the equation

$$p^2 = \frac{qc^2 s^2 (s^2-1)^2}{\rho a^4 (s^2+1)} \quad \ldots\ldots\ldots\ldots (47),$$

where $s = 2, 3, 4 \ldots$.

To find the period of the torsional vibrations, eliminate G_1 and H from the fourth of (45), and we get

$$\rho a^2 \ddot{\beta} = n\frac{d^2\beta}{d\phi^2} + \frac{2n+q}{2a}\frac{d^2v}{d\phi^2} - \tfrac{1}{2}q\beta \quad \ldots\ldots\ldots\ldots (48).$$

Eliminating G_1, H and N_2 from the third and fifth of (45), we get

$$-\frac{qc^2}{4a^4}\frac{d^4v}{d\phi^4} + \frac{c^2}{4a^3}(2n+q)\frac{d^2\beta}{d\phi^2} + \frac{nc^2}{2a^4}\frac{d^2v}{d\phi^2} + \frac{\rho c^2}{4a^2}\frac{d^2\ddot{v}}{d\phi^2} = \rho\ddot{v} \quad \ldots\ldots\ldots (49).$$

To solve these equations, let

$$\beta = Be^{i(pt+s\phi)},$$

$$v = Ve^{i(pt+s\phi)};$$

also let $\lambda = c/a$. Then

$$2a(\rho a^2p^2 - ns^2 - \tfrac{1}{2}q)B = (2n+q)s^2V,$$

and

$$\left\{\rho(1+\tfrac{1}{4}\lambda^2s^2)p^2 - \frac{\lambda^2s^2}{4a^2}(2n+qs^2)\right\}V = \frac{\lambda^2s^2}{4a}(2n+q)B;$$

whence

$$\rho^2a^2(1+\tfrac{1}{4}\lambda^2s^2)p^4 - \tfrac{1}{2}\rho\left\{2ns^2+q+\tfrac{1}{4}\lambda^2s^2\left[2n(s^2+2)+q(2s^2+1)\right]\right\}p^2 + \frac{nq\lambda^2s^2}{4n^2}(s^2-1)^2 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (50).$$

From this equation, we see that the torsional vibrations consist of two distinct types. One type consists of vibrations of short periods, which are comparable with the periods of the extensional vibrations; whilst the other consists of vibrations of long periods, which are comparable with those of the flexural vibrations. If we neglect powers of λ higher than the second, the value of frequency of the vibrations of long period is given by the equation

$$p_2^2 = \frac{nqc^2s^2(s^2-1)^2}{2\rho a^4(2ns^2+q)};$$

whence

$$\frac{p_2^2}{p_1^2} = \frac{2ns^2+2n}{2ns^2+q}.$$

Since $2n < q$, $p_2 < p_1$; whence the periods of the flexural vibrations are slightly less than those of the torsional vibrations of long period. The gravest note occurs when $s = 2$, in which case

$$p_2^2 = \frac{18nqc^2}{\rho a^4(8n+q)}.$$

16. When a naturally straight or twisted wire is subjected to a finite deformation, the displacements u, v, w are not small; and it would therefore be unsafe to apply the preceding theory to the finite bending and twisting of wires, without further investigation. There are, however, certain statical problems in which it is unnecessary to find the displacements; and in questions of this character equations (3) may be applied under certain limitations. The equations in question are of a perfectly general character, and apply to the finite bending of

wires, provided ρ and σ are the radii and torsion of the deformed wire; and, if the form of the axis of wire, when displaced, be given, ρ and σ are known quantities, and we may employ (3) to ascertain whether it is possible to maintain the wire in equilibrium by means of stresses applied to its extremities; and, when this is possible, the above equations will enable the stresses at any point of the wire to be calculated. If, however, the displaced form of the wire is not given, ρ and σ are unknown quantities, and (3) do not furnish sufficient equations to solve the problem of finite bending. But if the wire is a plane curve, both before and after deformation, and we assume that the flexural couple G_2 is proportional to the change of curvature, the problem of finite bending can be completely solved.

17. We shall now apply (3) to investigate the small deformation of a wire whose natural form is a helix, one end of which is fixed, whilst a weight W is attached to the other end; and we shall neglect the weight of the wire. This is the problem of an ordinary spring balance, which is used for weighing meat, and for a variety of other purposes.

In this case

$$\frac{1}{\rho} = \frac{\cos^2 a}{a}, \quad \frac{1}{\sigma} = \frac{\sin a \cos a}{a}.$$

Substituting the value of G_1, from the fourth of (3), in the last two, we obtain

$$\rho \frac{d^2 H}{ds^2} - \frac{G_2}{\sigma} + \frac{H}{\rho} = N_2 \quad \ldots\ldots\ldots\ldots\ldots\ldots (51),$$

$$\frac{dG_2}{ds} + \frac{\rho}{\sigma} \frac{dH}{ds} = -N_1 = \sigma \frac{dN_2}{ds} \quad \ldots\ldots\ldots\ldots (52),$$

whence

$$G_2 + \frac{\rho}{\sigma} H = \sigma (N_2 + W \cos a) \quad \ldots\ldots\ldots\ldots (53),$$

the constant being determined from the condition, that the couples must vanish at the end to which the weight is attached.

From (51) and (53), we obtain

$$N_2 = \tfrac{1}{2}\rho \left\{ \frac{d^2 H}{ds^2} - \frac{W}{\rho} \cos a + H \left(\frac{1}{\rho^2} + \frac{1}{\sigma^2} \right) \right\},$$

$$G_2 = \tfrac{1}{2}\rho\sigma \left\{ \frac{d^2 H}{ds^2} + \frac{W}{\rho} \cos a + H \left(\frac{1}{\rho^2} - \frac{1}{\sigma^2} \right) \right\},$$

$$-N_1 = \tfrac{1}{2}\rho\sigma \left\{ \frac{d^3 H}{ds^3} + \left(\frac{1}{\rho^2} + \frac{1}{\sigma^2} \right) \frac{dH}{ds} \right\}.$$

Substituting from (52) in the first of (3), we get

$$T = W\sin\alpha - \frac{G_2}{\rho} - \frac{H}{\sigma};$$

whence $$T = W\sin\alpha - \tfrac{1}{2}\sigma\left\{\frac{d^2H}{ds^2} + \frac{W}{\rho}\cos\alpha + H\left(\frac{1}{\rho^2} + \frac{1}{\sigma^2}\right)\right\}.$$

Accordingly, from the second of (3),

$$\frac{d^2}{ds^2}\left\{\frac{d^2H}{ds^2} + H\left(\frac{1}{\rho^2} + \frac{1}{\sigma^2}\right)\right\} + \left(\frac{1}{\rho^2} + \frac{1}{\sigma^2}\right)\left\{\frac{d^2H}{ds^2} + H\left(\frac{1}{\rho^2} + \frac{1}{\sigma^2}\right)\right\}$$
$$= \frac{W}{\rho}\left\{\frac{2\sin\alpha}{\rho\sigma} + \left(\frac{1}{\sigma^2} - \frac{1}{\rho^2}\right)\cos\alpha\right\}.$$

This differential equation determines H; and the four arbitrary constants are found from the conditions that, at the extremity to which the weight is attached,

$$H = 0,$$
$$G_1 = 0, \quad \text{whence} \quad \frac{dH}{ds} = 0,$$
$$G_2 = 0, \quad \text{whence} \quad \frac{d^2H}{ds^2} + \frac{W}{\rho}\cos\alpha = 0,$$
$$N_1 = 0, \quad \text{whence} \quad \frac{d^3H}{ds^3} = 0.$$

These conditions are sufficient for the complete solution of the problem, and we can thus obtain the values of the stresses at any point of the wire.

[*Added March 15th*, 1892.

18. I shall conclude this paper with some observations upon Saint-Venant's theory of the torsion of prisms.

The theory of wires which is associated with the names of Clebsch and Saint-Venant, depends upon the assumption, that the stresses P, Q, U are *accurately zero*; and this assumption appears to me to be a highly objectionable one to found a theory of wires upon. It is, of course, a legitimate subject of enquiry to start with the supposition that these stresses are zero, and then to proceed to ascertain what forces must be applied to produce this state of things; but an investigation of this kind is totally different from one, which seeks to base a theory upon an assumption of this character. The results of the paper show, that there are strong grounds for supposing that P, T and U are small quantities, *which may be treated as zero*, subject to the

limitations contained in the paper; and I consider the truth of this hypothesis to be as firmly established as is possible by an indirect method, which does not profess to be founded on a rigorous mathematical investigation, based upon the general equations of motion of an elastic solid. No valid reason exists for assuming that Q may be treated as zero; for, since it is not zero at the boundary, there is nothing to prevent its having a value, which would render it inadmissible to treat it as a small quantity.

19. I shall now proceed to examine Saint-Venant's theory of the torsion of prisms.

Let a cylinder, whose cross-section is any plane curve, and is bounded by two planes perpendicular to its axis, be maintained in equilibrium under the action of equal and opposite couples H. The conditions of equilibrium are the following (rectangular coordinates being used).

(i.) The resultant of the stresses, which act upon any cross-section, must be equal to the terminal couple G; whence

$$\left.\begin{aligned} \iint R\,dx\,dy &= 0, \\ \iint (Sx - Ty)\,dx\,dy &= H, \\ \iint T\,dx\,dy = 0, \quad \iint S\,dx\,dy &= 0, \\ \iint Rx\,dx\,dy = 0, \quad \iint Ry\,dx\,dy &= 0, \end{aligned}\right\} \quad \ldots\ldots\ldots\ (1),$$

where the integration extends over the cross-section.

(ii.) At every point of the curved boundary, the normal traction and the two tangential stresses perpendicular to the normal must vanish; whence, if θ be the angle which the normal at any point of the curved boundary makes with the axis of x,

$$\left.\begin{aligned} P\cos^2\theta + Q\sin^2\theta + U\sin 2\theta &= 0, \\ \tfrac{1}{2}(Q-P)\sin 2\theta + U\cos 2\theta &= 0, \\ S\sin\theta + T\cos\theta &= 0, \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ (2).$$

(iii.) At every point of the substance of the prism, it is necessary that the displacements should satisfy the general equations of equilibrium of an elastic solid.

To satisfy these conditions, Saint-Venant assumes that

$$u = -ryz, \quad v = rxz, \quad w = r\phi \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3),$$

where r is a constant, and ϕ is a function of x and y alone.

It can then be shown, that the general equations will be satisfied if

$$\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4).$$

Introducing the conjugate function ψ, such that

$$d\phi/dy = d\psi/dx, \quad d\phi/dx = -d\psi/dy,$$

the last of (2) requires that

$$\psi + \tfrac{1}{2}(x^2+y^2) = \text{const} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

along the boundary; whilst all the other conditions can also be shown to be satisfied. The solution is, therefore, reduced to determining the velocity potential of a liquid, which is contained in a rotating cylindrical cavity of the same form as the prism.

20. The merit of Saint-Venant's solution is that, by a happy process of induction, he arrived at a simple and elegant method of solving a problem, which it would be exceedingly difficult to attack by means of a general solution of the equations of equilibrium. The solution is, however, of a very special nature, and only applies to the case of a prism held in equilibrium by terminal couples. Moreover, the solution must not be pushed beyond due limits; for if a long, narrow bar of india-rubber, or a steel wire, were held in equilibrium in this manner, it is obvious that, if the couple exceeded a certain magnitude, which depends upon the linear dimensions and elastic constants of the material, the axis of the bar would be deformed, and Saint-Venant's solution would no longer be applicable. It must, however, be recollected, that the solution of statical problems is not necessarily unique, in the sense that a solution, which satisfies all the conditions, is the only one possible; for it frequently happens that when a system is in equilibrium under the action of given forces, there are several positions of equilibrium, some of which are stable, whilst others are unstable; and it might very well happen that Saint-Venant's solution gives a possible form of equilibrium, which becomes unstable unless certain conditions with regard to the linear dimensions and elastic constants are satisfied, and that a stable form of equilibrium exists, in which the state of things is altogether different from that contemplated in Saint-Venant's problem. It must also be

borne in mind, that the general equations of an elastic solid, *in terms of the displacements*, only apply to small deformations, and that a deformation involving finite displacements would require different treatment.

The assumption that P, Q, U are actually zero, or may, to a certain degree of approximation, be treated as zero, fortunately does not constitute a necessary part of the theory. It simply drops out as an incidental result. The essential feature of the solution consists in showing, that it is possible to satisfy all the conditions of the problem by certain *given* values of u and v, and a value of w which can be determined when the form of the cross-section is known.

21. The infirmity of Saint-Venant's solution is, that it does not completely take into account the conditions to be satisfied at the flat ends of the prism. If, for example, the flat ends were rigidly attached to two planes, one of which is fixed, whilst the other is turned through a small angle about the axis, the conditions which u and v must fulfil would be satisfied; but the value of w would have to be the same at every point of the flat ends. Hence the complete value of w must be some function of x, y, and z which becomes independent of x and y at the ends. If, however, the length of the prism is considerable, in comparison with the linear dimensions of its cross-section, and we confine our attention to points at some distance from the ends, the error committed, in neglecting the above-mentioned boundary conditions, would probably be unimportant.]

The Equations of Propagation of Disturbances in Gyrostatically Loaded Media, and of the Circular Polarization of Light. By J. LARMOR. Received November 10th, 1891. Read December 10th, 1891.

The object of the following analysis is to investigate the modification produced, in the characteristics of the propagation of undulations in a solid medium, by a distribution of angular momentum throughout the medium. We may, to make the problem physically realizable, imagine an elastic solid permeated by small spherical cavities, in each of which a fly-wheel is mounted, and is given rotating, with angular

velocity which is not itself affected by any motions of the medium, but which reacts against angular motions.

The chief interest of such an investigation is as an illustration of the effect of a magnetic field in rotating the plane of polarization of light. The conclusion drawn by Sir W. Thomson, that this phenomenon reveals the presence of rotation, or rather angular momentum, round the lines of magnetic force in the medium, is of deep significance, and has been one of the key-stones in the development of electrical theory by Clerk-Maxwell. It may also be of interest to compare a purely dynamical mode of treatment with the various hypotheses that have been in use to form a theory of the rotation of the plane of polarization of electrical waves, based on assumptions wholly electrical.

The hypothesis of gyrostatic cells interspersed throughout the medium, though at first sight artificial, is a correct realization of the current views of the influence of ponderable matter on the undulations of the aether. Any exhaustive optical investigation must take cognizance of the mutual influence of the two interpenetrating media, the aether and the ordinary matter. The treatment of the ordinary problem of undulatory propagation under these circumstances is rendered obscure and hypothetical. But when once the results of this inter-action are ascertained as experimental laws, the superposition of an angular momentum in the material medium does not require the introduction of any further hypothetical element in the analysis, and its effects may be unfolded by ordinary dynamics.

From the point of view of pure dynamical analysis, the problem is interesting, as forming an extension, which can be realized, of the dynamics of a continuous material system. Ordinary matter reacts against force simply by its mass or inertia, represented in the equations by a scalar coefficient; matter gyrostatically endowed possesses, in addition, another coefficient of inertia, of the vector or directed type, representing the gyrostatic momentum per unit volume, which may be referred to as its rotary inertia.

In the formation of the differential equations of an elastic solid in this way endowed, we shall follow the method of analysis which has been already illustrated in a discussion of the vibrations of a gyrostatically loaded chain.*

As a preliminary, it is interesting to remark, though not essential, that the disturbance in a strained medium, due to a small cell or cavity

* *Proc. Lond. Math. Soc.*, Vol. xxi., 1891, p. 423.

in it, is of a very local character, and falls off rapidly as the distance from the cavity increases; thus the presence of cavities, interspersed so thickly that their distances apart are even small multiples of their diameters, will not very much affect the elastic quality of the medium as a whole. But, if the distribution of angular momentum connected with these cells were so intense that the forces produced by its rotatory displacements were of the same order as the stresses in the medium, these local disturbances would then affect the gyrostatic forces, and at any rate alter the coefficients in the results of the analysis which follows. In the optical application, however, the gyrostatic forces which produce rotation are extremely small, compared with the stresses in the aether; we may, therefore, consider that the components of the rotation of the embedded gyrostats are the same as the average components of rotation for the elements of the medium in which they lie.

We shall find it convenient, in the first instance, to consider the medium divided up into right solid elements, of the type $\delta x\,\delta y\,\delta z$. We take uvw to represent the components of the elastic displacement at the point xyz, and LMN to represent the components of the angular momentum per unit volume near that point. The cubic dilatation δ and the components of the angular velocity ω_x, ω_y, ω_z of the medium are given by the equations

$$\delta = \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz},$$

$$\omega_x = \tfrac{1}{2}\frac{d}{dt}\left(\frac{dv}{dz} - \frac{dw}{dy}\right), \quad \omega_y = \tfrac{1}{2}\frac{d}{dt}\left(\frac{dw}{dx} - \frac{du}{dz}\right), \quad \omega_z = \tfrac{1}{2}\frac{d}{dt}\left(\frac{du}{dy} - \frac{dv}{dx}\right).$$

The components of the couples produced by these rotations, applied to this constant angular momentum, are, per unit volume,

$$2\mathfrak{L} = -M\omega_z + N\omega_y, \quad 2\mathfrak{M} = -N\omega_x + L\omega_z, \quad 2\mathfrak{N} = -L\omega_y + M\omega_x.$$

These will make themselves felt by additional shearing stresses acting over the surface of the element $\delta x\,\delta y\,\delta z$. In forming the equations of elastic displacement for the medium we are justified in replacing them by these shearing stresses. Thus, we may, so far as these rectangular elements are concerned, replace the couple $2\mathfrak{L}$ by either of the stresses

$$(YZ) = 2\mathfrak{L} \quad \text{or} \quad (ZY) = -2\mathfrak{L}$$

(or partly by one and partly by the other), where (YZ) denotes

tangential stress in the plane perpendicular to Y, which acts parallel to Z, and similarly for (ZY).

To eliminate this indeterminateness, and to find the actual distribution of this stress, we have to consider volume-elements of other forms; the consideration of an element of circular cross-section shows at once, by symmetry, that the component of this tangential stress round the circular section must be uniform, and therefore that $(YZ) = \mathfrak{L}$, $(ZY) = -\mathfrak{L}$.

We observe that when the gyrostatic quality is thus merged, and included in the elastic specification of the medium, the stress is no longer self-conjugate, in the sense that $(XY) = (YX)$, when the medium is in rotatory motion.

Denoting the elastic constants of an isotropic medium by λ and μ, so that $\lambda + \frac{2}{3}\mu$ is the compressibility, and μ is the rigidity, we have therefore the stress-components of the types

$$\left.\begin{aligned} (XX) &= \lambda\delta + 2\mu\frac{du}{dx}, & (YY) &= \ldots, \ (ZZ) = \ldots, \\ (YZ) &= \mu\left(\frac{dw}{dy}+\frac{dv}{dz}\right) + \mathfrak{L}, & (ZX) &= \ldots, \ (XY) = \ldots, \\ (ZY) &= \mu\left(\frac{dw}{dy}+\frac{dv}{dz}\right) - \mathfrak{L}, & (XZ) &= \ldots, \ (YX) = \ldots, \end{aligned}\right\}$$

wherein the shearing forces $2\mathfrak{L}$, $2\mathfrak{M}$, $2\mathfrak{N}$ are equally distributed between the tangential stresses, for the reason given above.

The dynamical equations are of the type

$$\frac{d(XX)}{dx} + \frac{d(YX)}{dy} + \frac{d(ZX)}{dz} = \rho\frac{d^2u}{dt^2},$$

which lead to

$$\left.\begin{aligned} (\lambda+\mu)\frac{d\delta}{dx} + \mu\nabla^2 u - \frac{d\mathfrak{N}}{dy} + \frac{d\mathfrak{M}}{dz} &= \rho\frac{d^2u}{dt^2}, \\ (\lambda+\mu)\frac{d\delta}{dy} + \mu\nabla^2 v - \frac{d\mathfrak{L}}{dz} + \frac{d\mathfrak{N}}{dx} &= \rho\frac{d^2v}{dt^2}, \\ (\lambda+\mu)\frac{d\delta}{dz} + \mu\nabla^2 w - \frac{d\mathfrak{M}}{dx} + \frac{d\mathfrak{L}}{dy} &= \rho\frac{d^2w}{dt^2}, \end{aligned}\right\}$$

in which $\mathfrak{L}\mathfrak{M}\mathfrak{N}$ have already been expressed in terms of uvw.

It is clear at once that compressional waves are unaffected by the gyrostatic momentum.

It will be sufficient for our present purpose to examine the case of [redacted] constant; though interesting mathematical problems are pre-

sented by other distributions of angular momentum—for example, radiating from a point or radiating from an axis.

When LMN are constant, we have

$$\frac{d\mathfrak{M}}{dz} - \frac{d\mathfrak{N}}{dy} = \tfrac{1}{2}L\vartheta - \tfrac{1}{2}G\frac{d\omega_x}{d\sigma},$$

where $$\vartheta \equiv \frac{d\omega_x}{dx} + \frac{d\omega_y}{dy} + \frac{d\omega_z}{dz} = 0,$$

and $$G\frac{d}{d\sigma} = L\frac{d}{dx} + M\frac{d}{dy} + N\frac{d}{dz}$$

Expressed in full, the equations of motion are therefore

$$\left.\begin{aligned}
(\lambda+\mu)\frac{d\delta}{dx} + \mu\nabla^2 u - \tfrac{1}{4}G\frac{d}{d\sigma}\frac{d}{dt}\left(\frac{dv}{dz} - \frac{dw}{dy}\right) &= \rho\frac{d^2u}{dt^2},\\
(\lambda+\mu)\frac{d\delta}{dy} + \mu\nabla^2 v - \tfrac{1}{4}G\frac{d}{d\sigma}\frac{d}{dt}\left(\frac{dw}{dx} - \frac{du}{dz}\right) &= \rho\frac{d^2v}{dt^2},\\
(\lambda+\mu)\frac{d\delta}{dz} + \mu\nabla^2 w - \tfrac{1}{4}G\frac{d}{d\sigma}\frac{d}{dt}\left(\frac{du}{dy} - \frac{dv}{dx}\right) &= \rho\frac{d^2w}{dt^2}.
\end{aligned}\right\}$$

The modification of the equations of a crystalline medium, produced by rotary inertia, is represented by the same new terms.

The analysis may now be shortened, by taking the axis of the momentum LMN as the axis of x; so that M and N will be zero, and

$$G\frac{d}{d\sigma} = L\frac{d}{dx}.$$

For a given wave travelling in the medium, we may take the plane of xy to contain the direction of propagation, so that d/dz is zero, and then

$$(\lambda+\mu)\frac{d\delta}{dx} + \mu\nabla^2 u - \tfrac{1}{4}L\frac{d^3w}{dx\,dy\,dt} = \rho\frac{d^2u}{dt^2},$$

$$(\lambda+\mu)\frac{d\delta}{dy} + \mu\nabla^2 v + \tfrac{1}{4}L\frac{d^3w}{dx^2\,dt} = \rho\frac{d^2v}{dt^2},$$

$$\mu\nabla^2 w - \tfrac{1}{4}L\frac{d^2}{dx\,dt}\left(\frac{du}{dy} - \frac{dv}{dx}\right) = \rho\frac{d^2w}{dt^2}.$$

Or, still shorter, we may retain the general equations, and consider a wave travelling with its front at right angles to the axis of x, so

that d/dy and d/dz are both zero; then

$$(\lambda+2\mu)\frac{d^2u}{dx^2}=\rho\frac{d^2u}{dt^2},$$

$$\left.\begin{aligned}\mu\frac{d^2v}{dx^2}-\tfrac{1}{4}L\frac{d^3w}{dx^2dt}&=\rho\frac{d^2v}{dt^2},\\ \mu\frac{d^2w}{dx^2}+\tfrac{1}{4}L\frac{d^3v}{dx^2dt}&=\rho\frac{d^2w}{dt^2}.\end{aligned}\right\}$$

Of these, the first equation exhibits the wave of normal displacement propagated with unaltered velocity. The other two represent the wave of tangential displacement propagated with a rotating plane of polarization,* the coefficient of rotation being proportional simply to the component angular momentum at right angles to its front.†

In the problem of reflection, whether for isotropic or crystalline media, the surface conditions are that the three components of displacement are continuous, and the two tangential stresses on the plane of reflection, as well as the normal stress at right angles to it, are continuous on both sides of the interface. Of these it is only the two tangential stresses that are affected by the gyratory terms; one of them is increased by $\mathfrak{L}$, and the other diminished by $\mathfrak{L}$, when the plane of reflection is that of yz, where

$$4\mathfrak{L}=\frac{d}{dt}\left\{-M\left(\frac{du}{dy}-\frac{dv}{dx}\right)+N\left(\frac{dw}{dx}-\frac{du}{dz}\right)\right\}.$$

Thus, if the axis of resultant momentum is at right angles to the interface, the surface conditions are not affected by it.

The difficulties in the way of the explanation of magnetic reflection of light, on these principles, are discussed in Sir W. Thomson's *Baltimore Lectures*, pp. 317–8. In the same work a compound gyrostatic molecule is described, which operates by change of internal configuration, and is susceptible to influence from translation instead of rotation; it does not require so great an angular velocity to produce in a small body the momentum necessary for finite effects. As this molecule has free periods of its own, it introduces dispersion, with respect both to refraction and rotation.

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* *Proc. Lond. Math. Soc.*, Vol. xxi., 1891. p. 429.

† The corresponding law, for magnetic reflection of light from metals, has been verified by Bois, *Phil. Mag.*, 1890, (1), p. 259.

There also arises the problem of the reflection, at the surface of a magnet, of the electric waves discovered by Hertz. This is a real question of actual phenomena, of which the solution may or may not correspond to the optical phenomena when the wave-length is made very small; though, if light waves run on all fours with the electric waves, we should expect a complete correspondence in this respect, owing to the smallness of the rotatory coefficient. The theory of electric propagation—whether as deduced by Faraday and Maxwell from the fundamental notion of an electric medium and the simplest formal type-equations that are consistent with ascertained phenomena, or, as arrived at in the method of Neumann and Helmholtz, by assuming the most general conceivable type of action at a distance between two electric systems and reducing to definiteness the formulæ so obtained by the aid of crucial experiments*—is, in its present form, not an ultimate dynamical theory. The dynamical conception of rotary inertia must therefore be replaced in it by some formal quality of rotational type, associated with the electric displacement; the verification of the existence of such a quality is a necessary part of the electric theory of light. As is well known, a rotational quality of this kind relating to electric currents in a conducting medium has been detected by Hall; and its hypothetical extension to electric displacements in an insulating medium has been applied to the theory of electric waves by Rowland. The equations of propagation are of the same formal type as those given above. But when we come to consider the question of reflection, there is room for further variety of hypothesis. This problem has in fact been recently treated by Mr. Basset,† on the hypothesis that the total electric force consists of the electro-static and the electro-kinetic parts which fit in with Maxwell's scheme, together with an additional though small part depending for its origin on an external magnetic force, which cannot be said to have a natural place in an ideal simple scheme of electro-dynamic relations, but must rather be classed as a residual phenomenon of a higher order, as is the rotational property of quartz in the elastic solid theory of optics. It will be seen, on comparison, that the boundary conditions found in that way are not the same as those here obtained by referring the phenomena to an ultimate dynamical theory, in which it is essential that the stress shall not be discontinuous in crossing an interface. Whether the gyratory coefficient is due to a single rotator,

* *Cf. Proc. Roy. Soc.*, Vol. XLIX., 1891, p. 521.
† *Phil. Trans*, 1891.

as taken above, or to a more complicated rotating system, the equations of propagation are the same in form,—at any rate, so far as they depend only on the angular momentum of the rotator, and are not sensibly influenced by its free periods. The gyratory terms in them, as they involve the time to a lower negative degree than the terms due to ordinary inertia, must, for purposes of boundary conditions, be considered as derived from a bodily stress, and not classed with the inertia terms; for the latter procedure would involve infinite accelerations at the interface. If the relations were deduced, in Green's manner, from an expression for the energy of the medium, this latter procedure would involve a distribution of energy over the interface, of finite surface density (that is, infinite volume density at the surface), which would take part in the propagation of the waves; it may, however, be possible to express the surface-integral representing this distribution as a volume-integral, and so refer it to the medium as a whole. There is thus room for different types of boundary conditions, depending on the structure of the gyrostatic elements. In the case of compound elements there would also occur the anomalous rotatory dispersion corresponding to the free periods of the elements; and this would affect the equations of propagation as well as the boundary conditions.

This leads to, and is proved by, another remark, with reference to the three types of term, originally suggested by Sir G. B. Airy as competent to render an account of the optical magnetic rotation.* The case considered by him being that of symmetry round the axis of z, the equation of propagation may be expressed in the form

$$\rho \frac{d^2\theta}{dt^2} = k \frac{d^2\theta}{dz^2} + P,$$

where θ represents $u + v\sqrt{-1}$, and P may have any of the forms

$$\kappa_1 \frac{d^3\theta}{dz^2 dt}, \quad \kappa_2 \frac{d^3\theta}{dt^3}, \quad \kappa_3 \frac{d\theta}{dt},$$

or may include them all.

Now the dimensions of κ_1/ρ, κ_2/ρ, κ_3/ρ, considered as physical quantities involved in this equation, are $[L^2T^{-1}]$, $[T]$, and $[T^{-1}]$. Thus the first of these forms must involve a distribution of angular momentum throughout the volume; while the second and third could only be

* Maxwell, *Electricity and Magnetism*, Vol. II., § 830.

derived from theories which involve the introduction of an absolute time-constant, such, for example, as a period of free molecular vibration, thus verifying the statement of the last paragraph. The somewhat exact correspondence between Verdet's experiments*, on the relation of magnetic rotation to dispersion, for a substance of simple chemical structure, like bisulphide of carbon, and the law deducible from the first of the three types of equations, is in accord with these considerations.

In the same way the *reversible* rotatory property of quartz, and sugar solutions, requires an additional term in the equation of motion of a wave parallel to z, of one of the types

$$\frac{d^3\theta}{dz\,dt^2} \quad \text{and} \quad \frac{d^3\theta}{dz^3};$$

where $\theta = u + v\sqrt{-1}$. The first of these represents a coefficient of inertia of the type $\rho + \sigma \frac{d}{dz}$, *i.e.*, one which, in the wave motion considered, has a varying term which is a harmonic function of z. The second connotes a law of elasticity involving differential coefficients of odd order of the displacement. Of these terms the latter is thus the only one that can be derived from an elastic or other statical theory based on a symmetry in the medium; while the former would require some sort of motional structure (*i.e.*, the existence of steady forces arising from the inertia opposing unrecognised steady motions) to justify its adoption.

On the Contacts of Systems of Circles. *By* ALEXANDER LARMOR, M.A., Fellow of Clare College, Cambridge. Communicated in abstract, November 12th, 1891. Received 29th February, 1892.

In his paper in the *Proceedings of the Royal Irish Academy*, Vol. IX., Part IV., Dr. Casey gave, in a very elegant form, as a relation among the lengths of their common tangents, the condition that four circles should touch the same circle. His discussion of this condition is by no

* Maxwell, *loc. cit.*

means complete, and, in the subsequent portion of the paper, after a proof that the condition is *necessary*, it is assumed to be *sufficient*.

The condition is now well known, and has been incorporated in text-books of geometry, and discussed by various writers; but I am not aware that anything substantial has been added to Casey's original treatment.

I have attempted to give a somewhat exhaustive discussion of this condition, whether we look upon it as a relation among the common tangents of the given circles, or as a relation among their angles of intersection, and have shown, by purely geometrical reasoning, that it is *sufficient* as well as *necessary*—that it is not satisfied by any relation between the circles except this contact relation.

The condition is then applied to the investigation of the contact relations of the eight circles which touch three given circles (due to Casey), and also of the eight circles each of which passes through three of the points of intersection of three given circles.* The latter case is also derived from the former by means of the polar transformation on the sphere.

From these results are derived the contact relations of the thirty-two right circular cones which touch three given covertical right circular cones, and of the thirty-two right circular cones which pass through three of the twelve lines of intersection of three given covertical right circular cones. Casey (*loc. cit.*) has shown that the former system of thirty-two right circular cones can be divided into a certain number of groups of four, each of which touches another right circular cone. By means of a transformation I have shown that a relation subsists for double this number of groups.

From these theorems are then derived, by the theory of projections, the contact relations of the thirty-two conics having double contact with a given conic, and touching three other conics having double contact with this given conic; and also of the thirty-two conics having double contact with a given conic and passing through three of the twelve points of intersection of three given conics having double contact with this one. The former system has been discussed analytically by Casey, but he has only detected half the full number of groups which possess the property under consideration.

Several important particular cases of these general theorems are also noticed.

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* Communicated in abstract to Section A of the British Association, 1887.

We begin the discussion with the explanation of the nomenclature adopted, and the statement of some Lemmas which will be required in the course of the paper.

The small circle on a sphere, traced out by that pole of a tangent great circle, to a given small circle, which lies on the opposite side of the great circle, will be termed the polar of the given small circle, and the small circle traced out by the other pole will be termed the antipolar.

Two circles on a sphere will be said to have *internal* contact when they lie on the same side of their tangent great circle at their point of contact, and to have *external* contact when they lie on opposite sides of it.

It is important to remark that the geometry on a sphere is more symmetrical and admits of wider transformation than geometry on a plane. The process of inversion by reciprocal radii vectores on a plane corresponds to simple projection from a point, of the diagram on a sphere, and in this connexion its fundamental character, as regards similarity of small parts in a diagram and its inverse, appears in its true light. We have, in addition, a symmetrical method of polarizing on a sphere, wherein each point is replaced by its polar great circle. But this increased facility of transformation introduces its corresponding difficulties. A great circle has, in fact, two poles, so that any diagram has two polar diagrams, which must be discriminated. It is this circumstance which constitutes the chief obstacle to the general application of the method. It may be remarked that there is an analogous difficulty in the coordinate geometry of right circular cones. Thus, for example, the discussion given in Salmon's *Solid Geometry*, Art. 259 *seq.*, of Sir Andrew Hart's contact theorem relating to circles on a sphere, is not valid. The demonstrations apply to cones, not to circles; each cone meets the sphere in *two* opposite circles, and there is no means given of discriminating between them. The same difficulty occurs throughout in the treatment of small circles on a sphere by means of Cartesian equations.

It will be found that, for a transformation involving several contacts, the discrimination between the polar and antipolar diagrams, although at first sight it seems a matter of great complexity, may be effected by the use of the two following principles:—

(a) If two circles have internal contact with each other, their polars have internal contact with each other (and their antipolars have internal contact with each other).

(β) If two circles have external contact with each other, the polar of either has external contact with the antipolar of the other.

The Lemmas which we shall have occasion to use are the following:—

(i.) If any point and its polar plane with regard to a quadric be taken as centre and plane of projection, any plane section of the quadric projects into a conic having double contact with the section of the quadric by the plane of projection. If any other plane be taken as plane of projection, any plane section of the quadric projects into a conic having double contact with the section of the tangent cone by that plane.

When the plane section of the quadric intersects its section by the polar plane in real points, this proposition is evident; and therefore, by the principle of continuity, it holds generally.

(ii.) If two circles, on a sphere or *in plano*, are the inverses of two others with regard to a given point, they will all four touch each of a group of four circles which have a common radical centre situated at that point.

Consider the general case in which the circles are plane sections of a sphere. The two circles and their respective inverses lie on two cones having their common vertex at the given point. These cones have, in general, four common tangent planes, which pass through their common vertex and cut the sphere in four circles touching the two given circles and their inverses.

(iii.) Given three circles 1, 2, 3 and a point P, there is only one other point Q from which the tangents drawn to the three given circles are proportional to those from P, viz., the other point of concourse of the three circles passing through P and coaxal with 1, 2; 2, 3; 3, 1, respectively.

The straight line PQ passes through the radical centre of the circles 1, 2, 3, and it easily follows that P and Q are inverses with regard to the circle which cuts them orthogonally. The eight tangent circles of 1, 2, 3 consist of four inverse pairs with regard to their orthogonal circle.

Hence, if P is on neither circle of a given inverse pair, Q is not on either.

We now proceed, in effect, to discuss, and in some measure to systematize, the very remarkable complex of contacts which bind together the whole system of plane sections of a sphere, or quadric

surface, that are constructed on the basis of any three plane sections (and their three opposites) by drawing in succession other plane sections themselves determined solely by conditions of contact.

1. If any four circles 1, 2, 3, 4 touch the same circle, a relation of the form

$$12 \cdot 34 + 14 \cdot 23 - 13 \cdot 24 = 0$$

subsists among their common tangents, the common tangents involved being direct or transverse according as the contacts of the corresponding circles with the touched circle are of the same or opposite nature.

Where it is necessary, in the following analysis, to distinguish between direct and transverse common tangents, their lengths will be denoted by symbols of the types 12 and (12) respectively.

Let A, B (Fig. 1) be the centres of two circles, O that of a circle

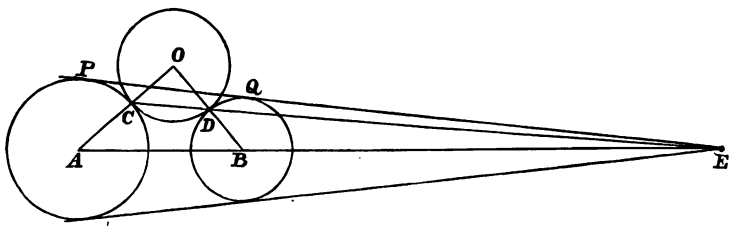

FIG. 1.

touching them both externally or both internally at C, D respectively; PQ a direct common tangent, and E the external centre of similitude.

Let ρ be the radius of the touching circle. Then, since ODB is a transversal drawn across the triangle ACE,

$$\frac{AO}{\rho} = \frac{AB}{BE} \cdot \frac{ED}{DC};$$

and, since OCA is a transversal drawn across the triangle EBD,

$$\frac{BO}{\rho} = \frac{AB}{AE} \cdot \frac{EC}{CD}.$$

Hence $$\frac{AO.BO}{\rho^2} = \frac{EC.ED}{AE.BE} \cdot \frac{AB^2}{CD^2} = \frac{EP.EQ}{AE.BE} \cdot \frac{AB^2}{CD^2} = \frac{t^2}{CD^2};$$

therefore $$\frac{t}{CD} = \frac{\sqrt{AO.BO}}{\rho},$$

where t is the length of the external common tangent PQ.

When the contacts are one external and one internal, if we make use of the corresponding Fig. 2, in which I the internal centre of similitude takes the place of E, we are led to this same relation; where t is now the length of the transverse common tangent.

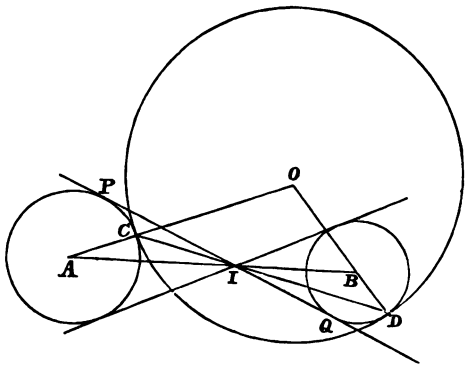

FIG. 2.

Let P_1, P_2, P_3, P_4 be the points of contact of the circles with their common tangent circle (Fig. 3); then, by Ptolemy's theorem,

$$P_1P_2 . P_3P_4 + P_1P_4 . P_2P_3 - P_1P_3 . P_2P_4 = 0.$$

Hence, denoting by 12 the common tangent of the circles 1 and 2,

$$12 . 34 + 14 . 23 - 13 . 24 = 0,$$

where the common tangents are direct or transverse according as the contacts with the touched circle are of the same or of opposite nature.

It is to be observed that the six common tangents involved must be either—

(i.) all direct,

(ii.) three direct and three transverse,

(iii.) two direct and four transverse;

and that in the two latter cases their distribution must be according

to the types—

$$(12).\ 34\ +(14).\ 23\ -(13).24=0,$$

and

$$(12).(34)+(14).(23)-\ 13\ .24=0,$$

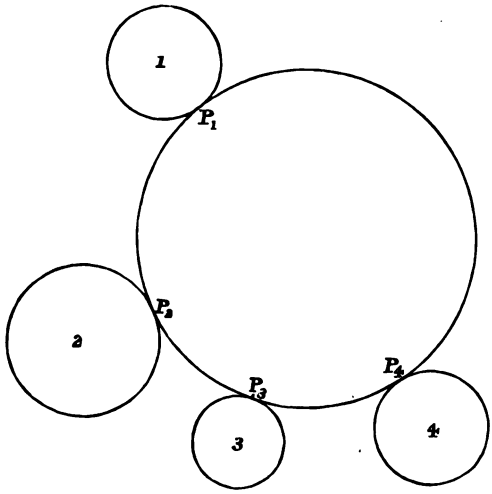

FIG. 3.

respectively, where, as defined above, (12) denotes the transverse common tangent of 1 and 2.

We also remark that the product affected with the negative sign is that of which the corresponding chords of contact intersect without being produced.

2. If the circle 4 reduce to a point, we see that

$$12.34+14.23-13.24=0,$$

$$-12.34+14.23+13.24=0,$$

or

$$12.34-14.23+13.24=0,$$

according as the point 4 is on the arc 13, 12, or 23, respectively, of either of the circles which touch 1, 2, 3 all internally or all externally.

If it be on either circle which has contacts of similar nature with 1 and 3, and of the opposite nature with 2, then (Fig. 4)

$$(12).(34)+(14).(23)-13.24=0,$$

$$-(12).(34)+(14).(23)+13.24=0,$$

or

$$(12).(34)-(14).(23)+13.24=0.$$

If it be on either circle which has contacts of similar nature with 1 and 2, and of the opposite nature with 3, then (Fig. 5)

$$12.(34)+14.(23)-(13).24=0,$$
$$-12.(34)-14.(23)+(13).24=0,$$
or $$12.(34)-14.(23)+(13).24=0.$$

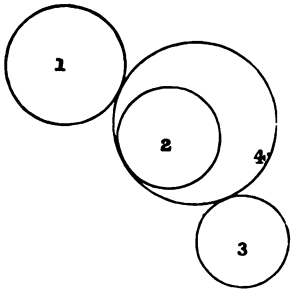

FIG. 4.

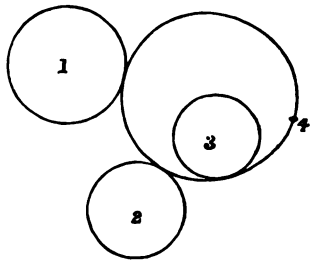

FIG. 5.

If it be on either circle which has contacts of similar nature with 2 and 3, and of the opposite nature with 1, then (Fig. 6)

$$(12).34+(14).23-(13).24=0,$$
$$-(12).34+(14).23+(13).24=0,$$
or $$(12).34-(14).23+(13).24=0.$$

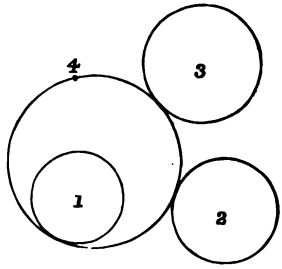

FIG. 6.

These alternatives hold in each case according as the point 4 is on the arc 13, 12, or 23, respectively, it being regarded as a point circle lying on the same side of the tangent circle as the circle 2.

3. Conversely, if any one of the relations which occur in the last

Art. subsist among the mutual common tangents of the circles 1, 2, 3, and the point 4, that point must lie on one or other of the two arcs of the pair of tangent circles of 1, 2, 3 for which that particular relation has here been proved to subsist.

Suppose (Fig. 7) that P is a position of the point 4, *on neither of the arcs* 13 of the pair of tangent circles for which the relation

$$12.34+14.23-13.24=0$$

is satisfied.

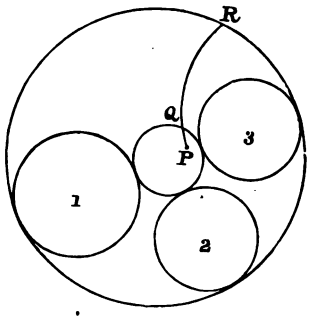

FIG. 7.

Through P describe a circle coaxal with 1 and 3, and let it cut either of these arcs in Q.

Then $12.3P+23.1P-13.2P=0$, by hypothesis,

and $12.3Q+23.1Q-13.2Q=0$, by Art. 2;

also $3P : 1P :: 3Q : 1Q$,

since P and Q are on a circle coaxal with 1 and 3.

Hence $3P : 2P : 1P :: 3Q : 2Q : 1Q$,

which is impossible, by Lemma iii.

4. If a relation of the form

$$12\ .\ 34 \pm 14\ .\ 23 \pm 13\ .24=0,$$

$$(12).\ 34 \pm (14).\ 23 \pm (13).24=0,$$

or $$(12).(34) \pm (14).(23) \pm 13\ .24=0,$$

subsist among the common tangents of four circles 1, 2, 3, 4, they will have a common tangent circle.

Take that circle, say 4, whose radius is not greater than that of any of the three remaining circles. With the centre of each of the

remaining circles as centre describe a circle, whose radius is equal to the sum or difference of its radius and that of the circle 4, according as the common tangent of it and 4 is transverse or direct.

These three new circles 1′, 2′, 3′, together with the centre of the circle 4 (a point circle) form a group of four circles having the same mutual common tangents as the four given circles, so that the given relation is satisfied for this system, and it follows by Art. (3) that the centre of the circle 4 lies on one or other of a pair of common tangent circles of 1′, 2′, 3′, and, hence, that 4 touches one or other of a pair of common tangent circles of 1, 2, 3.*

5. To express the common tangents of a pair of circles in terms of their radii and angle of intersection.

Let t, t' be the lengths of their direct and transverse common tangents,

δ the distance between their centres,

θ their angle of intersection (viz., the one which lies inside both).

Then, for circles *in plano* (Fig. 8),

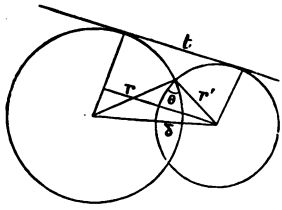

FIG. 8.

$$\begin{aligned}\delta^2 &= r^2+r'^2+2rr'\cos\theta\\ &= t^2+r^2+r'^2-2rr'\\ &= t'^2+r^2+r'^2+2rr'.\end{aligned}$$

Hence
$$\left.\begin{aligned}t^2 &= 2rr'(1+\cos\theta),\\ t'^2 &= -2rr'(1-\cos\theta),\end{aligned}\right\}$$

or
$$\left.\begin{aligned}t &= 2\sqrt{rr'}\cdot\cos\frac{\theta}{2} \qquad \text{.....(i.),}\\ t' &= 2\sqrt{-1}\cdot\sqrt{rr'}\cdot\sin\frac{\theta}{2} \qquad \text{.....(ii.).}\end{aligned}\right\}$$

* It will touch both circles of the pair if, in addition to the given relation, the circles 1, 2, 3, 4 have a common orthogonal circle.

For circles on a sphere (Figs. 9, 10),

$$\begin{aligned}\cos\delta &= \cos r\cos r' - \sin r\sin r'\cos\theta\\ &= \sin r\sin r' + \cos r\cos r'\cos t\\ &\qquad\qquad \text{from triangle } OO'P \text{ (Fig. 9)}\\ &= -\sin r\sin r' + \cos r\cos r'\cos t'\\ &\qquad\qquad \text{from triangle } OO'P' \text{ (Fig. 10).}\end{aligned}$$

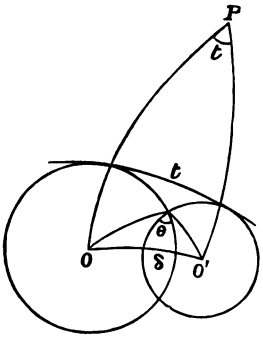

FIG. 9.

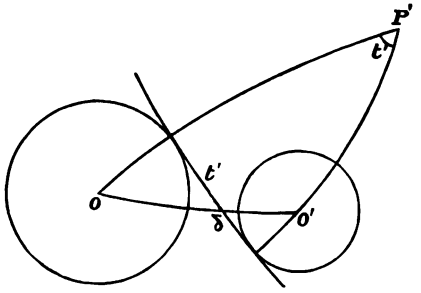

FIG. 10.

Hence
$$\left.\begin{aligned}\cos r\cos r'(1-\cos t) &= \sin r\sin r'(1+\cos\theta),\\ \cos r\cos r'(1-\cos t') &= -\sin r\sin r'(1-\cos\theta),\end{aligned}\right\}$$

or
$$\left.\begin{aligned}\sin\frac{t}{2} &= \sqrt{\tan r\tan r'}\cos\frac{\theta}{2} \qquad \text{.....(i.),}\\ \sin\frac{t'}{2} &= \sqrt{-1}\tan r\tan r'\sin\frac{\theta}{2} \qquad \text{.....(ii.).}\end{aligned}\right\}$$

6. From Arts. (4) and (5), we see that the necessary and sufficient conditions that four circles *in plano* should have a common tangential circle is, when expressed in terms of their angles of intersection,

$$\cos\frac{12}{2}\cos\frac{34}{2} \pm \cos\frac{14}{2}\cos\frac{23}{2} \pm \cos\frac{13}{2}\cos\frac{24}{2} = 0,$$

$$\sin\frac{12}{2}\cos\frac{34}{2} \pm \sin\frac{14}{2}\cos\frac{23}{2} \pm \sin\frac{13}{2}\cos\frac{24}{2} = 0,$$

or
$$\sin\frac{12}{2}\sin\frac{34}{2} \pm \sin\frac{14}{2}\sin\frac{23}{2} \pm \cos\frac{13}{2}\cos\frac{24}{2} = 0.$$

By inversion it is clear that this also holds for four circles on a

sphere, and, in this latter case, the condition, when expressed in terms of the common tangents of the circles, is

$$\sin\frac{12}{2}\sin\frac{34}{2} \pm \sin\frac{14}{2}\sin\frac{23}{2} \pm \sin\frac{13}{2}\sin\frac{24}{2} = 0,$$

$$\sin\frac{(12)}{2}\sin\frac{34}{2} \pm \sin\frac{(14)}{2}\sin\frac{23}{2} \pm \sin\frac{(13)}{2}\sin\frac{24}{2} = 0,$$

or $$\sin\frac{(12)}{2}\sin\frac{(34)}{2} \pm \sin\frac{(14)}{2}\sin\frac{(23)}{2} \pm \sin\frac{13}{2}\sin\frac{24}{2} = 0.$$

These conditions, in the case of four circles on a sphere, may also be established directly, by following a method precisely analogous to that adopted in the case of four circles *in plano*.

[I am indebted to Mr. Lachlan for the following alternative proof of the condition, directly applicable to the case in which the three circles intersect each other in real angles.

Let X_1, X_2, X_3, X_4 be four circles whose *external* angles of intersection are connected by the relation

$$\sin\tfrac{1}{2}\omega_{12}\,.\,\sin\tfrac{1}{2}\omega_{34} + \sin\tfrac{1}{2}\omega_{14}\,.\,\sin\tfrac{1}{2}\omega_{23} - \sin\tfrac{1}{2}\omega_{13}\,.\,\sin\tfrac{1}{2}\omega_{24} = 0.$$

Let Y and Y' be two circles which touch X_1, X_2, X_3 in the same sense, *i.e.*, all internally or all externally; then either Y or Y' touches X_4 in the same sense as the others.

Now two circles can be drawn to cut X_2, X_3 at the same angles as X_4 cuts them, and at the same time to touch Y in the same sense as Y touches X_2, X_3. Let these circles be X_5, X_6. It is easy to see that, if P_5, P_6 be the points of contact of these with Y, P_5 and P_6 will lie on opposite arcs of Y which join the points of contact X_2, X_3. Hence, if P_5 be on the same arc as the point of contact of X_1,

$$\sin\tfrac{1}{2}\omega_{12}\,.\,\sin\tfrac{1}{2}\omega_{35} + \sin\tfrac{1}{2}\omega_{15}\,.\,\sin\tfrac{1}{2}\omega_{23} - \sin\tfrac{1}{2}\omega_{13}\,.\,\sin\tfrac{1}{2}\omega_{25} = 0,$$

or $$-\sin\tfrac{1}{2}\omega_{12}\,.\,\sin\tfrac{1}{2}\omega_{35} + \sin\tfrac{1}{2}\omega_{15}\,.\,\sin\tfrac{1}{2}\omega_{23} + \sin\tfrac{1}{2}\omega_{13}\,.\,\sin\tfrac{1}{2}\omega_{25} = 0,$$

according as P_5 is on the arc joining the points of contact of X_1, X_3, or on that joining the points of contact of X_1, X_2.

Hence, since

$$\sin\tfrac{1}{2}\omega_{25} = \sin\tfrac{1}{2}\omega_{24}, \quad \text{and} \quad \sin\tfrac{1}{2}\omega_{35} = \sin\tfrac{1}{2}\omega_{34},$$

we must have $$\sin\tfrac{1}{2}\omega_{15} = \pm\sin\tfrac{1}{2}\omega_{14};$$

the lower sign is impossible, since all the angles are less than two right angles.

In the same manner there will be one circle, X_7 say, which touches Y' in the same sense as Y' touches X_2, X_3, such that

$$\sin \tfrac{1}{2}\omega_{27} = \sin \tfrac{1}{2}\omega_{24}, \quad \text{and} \quad \sin \tfrac{1}{2}\omega_{37} = \sin \tfrac{1}{2}\omega_{34},$$

and therefore $$\sin \tfrac{1}{2}\omega_{17} = \sin \tfrac{1}{2}\omega_{14}.$$

Hence $$\omega_{15} = \omega_{17} = \omega_{14} \quad \text{or} \quad 2\pi - \omega_{14};$$

the latter alternative is impossible, since $2\pi - \omega_{14} > \pi$.

Hence the circles X_5, X_7 cut X_1, X_2, X_3 at the same angles as X_4 cuts them.

Let X be the orthogonal circle of X_1, X_2, X_3, and let X_4' be the inverse of X_4 with respect to X; then it follows that X_5 and X_7 must coincide with X_4 and X_4'. That is to say, one of the pair X_5, X_7 must coincide with X_4.

Hence X_4 touches Y or Y'. That is to say X_1, X_2, X_3, X_4 touch another circle, each in the same sense.*]

7. The eight circles which can be drawn to touch three given circles on a sphere, or *in plano*, can be divided into fourteen groups of four, each of which touches another circle (Hart and Casey).

Let us assume that the three given circles intersect, as in (Fig. 11).

The three given circles divide the sphere into eight triangles. Consider any one of these triangles, and the four circles related to it, as the inscribed and escribed circles of a plane triangle.

Then, since these four circles, 1, 2, 3, 4, say, touch the three given circles as a, b, c, d touch α, β, γ in Fig. 11, the following relations

---

* It is interesting to observe that when X_4 cuts X_1, X_2, X_3 at equal angles, the condition becomes

$$\sin \tfrac{1}{2}\omega_{12} + \sin \tfrac{1}{2}\omega_{23} - \sin \tfrac{1}{2}\omega_{13} = 0,$$

the interpretation of which is that Y, Y' and X_4 touch each other at the same point. For, by inverting X_1, X_2, X_3 into three equal circles, this condition reduces to the relation

$$t_{12} + t_{23} - t_{13} = 0,$$

among the common tangents, which shows that these three circles touch two parallel straight lines, and it is clear that X_4 inverts into a straight line parallel to them. Hence, in the original figure, Y, Y' and X_4 touch each other at the origin of inversion.—A. L.

subsist among their common tangents, viz.,

$$\sin\frac{14}{2}\sin\frac{(23)}{2}+\sin\frac{24}{2}\sin\frac{(13)}{2}-\sin\frac{12}{2}\sin\frac{(34)}{2}=0,$$

$$\sin\frac{13}{2}\sin\frac{(24)}{2}+\sin\frac{(14)}{2}\sin\frac{23}{2}-\sin\frac{(34)}{2}\sin\frac{12}{2}=0,$$

$$\sin\frac{(12)}{2}\sin\frac{34}{2}+\sin\frac{14}{2}\sin\frac{(23)}{2}-\sin\frac{13}{2}\sin\frac{(24)}{2}=0.$$

Hence $$\sin\frac{(14)}{2}\sin\frac{23}{2}+\sin\frac{(12)}{2}\sin\frac{34}{2}-\sin\frac{(13)}{2}\sin\frac{24}{2}=0,$$

showing that the circles 1, 2, 3, 4 touch another circle, the contacts with 2, 3, 4 being of similar nature, and the contact with 1 of the opposite nature.

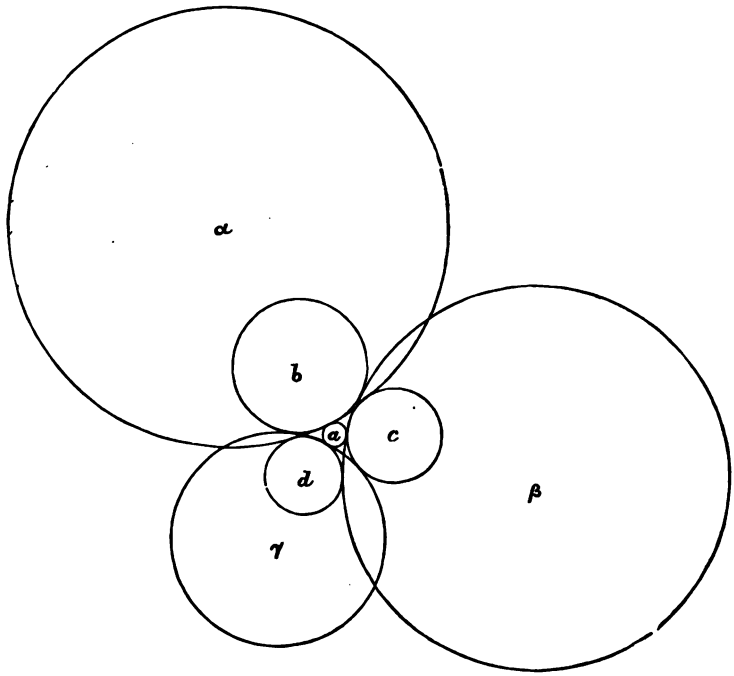

Fig. 11.

As there are eight triangles, there are eight groups of this species. Again, the eight circles which touch three given circles on a sphere

clearly consist of 1, 2, 3, 4, and their inverses 1′, 2′, 3′, 4′ with regard to the point of intersection of the planes of the three given circles.

The six groups of four, viz.,

$$11'22',\ 11'33',\ 11'44',\ 22'33',\ 22'44',\ 33'44',$$

consist each of two circles and their inverses with regard to this point, and therefore (by Lemma ii.) are each tangential to four circles, *i.e.*, to another circle in addition to the three given circles.

8. The eight circles which can be drawn through the points of intersection of three given circles, taken three by three, can be divided into eight groups of four, possessing the property that each circle of any group touches *four* other circles.

Let us denote the points of intersection of the three given circles by A, B, C, A', B', C', respectively, and the point of intersection of their planes by O.

The eight circles in question are those circumscribing the triangles

$$\begin{aligned} &ABC,\ A'B'C',\\ &ABC',\ A'B'C,\\ &AB'C,\ A'BC',\\ &A'BC,\ AB'C'. \end{aligned}$$

Consider the six groups of four :—

$$\begin{array}{llllll} ABC, & ABC', & AB'C, & AB'C', & AB'C', & ABC,\\ AB'C', & AB'C, & ABC, & AB'C, & ABC', & ABC',\\ A'B'C', & A'B'C, & A'BC', & A'BC, & A'BC, & AB'C',\\ A'BC, & A'BC', & A'B'C', & A'BC, & A'B'C, & A'B'C. \end{array}$$

Each of these groups consists of two circles and their inverses with regard to the point O.

Hence (Lemma ii.) the four constituents of each group touch each of a group of four circles.

Consider the two groups :—

$$\begin{aligned} &ABC,\ A'B'C',\\ &AB'C',\ A'BC,\\ &A'BC',\ AB'C,\\ &A'B'C,\ ABC'. \end{aligned}$$

The angles of intersection of the circles in the former group are given by the following scheme (as may easily be seen from the figure), in which α, β, γ denote the angles of intersection of the three given circles,

	ABC	$AB'C'$	$A'BC'$	$A'B'C$
ABC		$\beta \sim \gamma$	$\gamma \sim \alpha$	$\alpha \sim \beta$
$AB'C'$	$\beta \sim \gamma$		$\alpha+\beta-\pi$	$\alpha+\gamma-\pi$
$A'BC'$	$\gamma \sim \alpha$	$\alpha+\beta-\pi$		$\beta+\gamma-\pi$
$A'B'C$	$\alpha \sim \beta$	$\alpha+\gamma-\pi$	$\beta+\gamma-\pi$	

and, if we call the circles, respectively, 1, 2, 3, 4, and suppose $\alpha > \beta > \gamma$, it is easily verified that their angles of intersection satisfy the following four relations :—

$$\sin\frac{14}{2}\cos\frac{23}{2}+\sin\frac{12}{2}\cos\frac{34}{2}-\sin\frac{13}{2}\cos\frac{24}{2}=0,$$

$$\sin\frac{12}{2}\cos\frac{34}{2}-\sin\frac{23}{2}\cos\frac{14}{2}+\sin\frac{24}{2}\cos\frac{13}{2}=0,$$

$$-\sin\frac{24}{2}\cos\frac{13}{2}+\sin\frac{14}{2}\cos\frac{23}{2}+\sin\frac{34}{2}\cos\frac{12}{2}=0,$$

$$-\sin\frac{23}{2}\cos\frac{14}{2}+\sin\frac{13}{2}\cos\frac{24}{2}+\sin\frac{34}{2}\cos\frac{12}{2}=0,$$

whence (Art. 6) the circles 1, 2, 3, 4 touch four other circles.

In the same manner, or by inversion with regard to the centre of the sphere, it may be proved that the latter group

$$A'B'C',$$
$$A'BC,$$
$$AB'C,$$
$$ABC',$$

touch four other circles.

9. The contact relations of the system of eight circles which pass through the points of intersection of three given circles can also, in the two latter cases, be derived, by the following transformation, from those of the system of eight circles which touch three given circles.

Consider the figure consisting of three great circles on a sphere, and their eight tangential circles. Each great circle has two poles. Let A, B, C be the poles of the three given great circles, respectively, which do not lie on the same side as the inscribed circle of the spherical triangle formed by them, and let A', B', C' be their three remaining poles.

The polar figure corresponding to the three great circles, and their eight tangential circles, consists of three great circles intersecting in A, A', B, B', C, C', respectively, and the eight circles passing through these points three by three.

The inscribed and three escribed circles of the original spherical triangle touch a circle (Sir A. Hart's circle) the nature of the contacts being, in the case we have chosen (Fig. 11), the first internal and the last three external. Hence (Lemmas α, β) the polar of Hart's circle touches the polar of the first internally, and the antipolars of the other three externally, *i.e.*, the four circles passing through the points

$$ABC,$$
$$A'B'C,$$
$$A'BC',$$
$$AB'C'.$$

Now we observe that the original spherical triangle has three co-lunar triangles, and that each of these has a Hart circle of its own; hence, in the polar figure, we see that the four circles passing through the points

$$ABC',$$
$$A'BC,$$
$$AB'C,$$
$$A'B'C',$$

touch the polar of the Hart circle of a co-lunar triangle; and, in

ing with regard to the centre of the sphere, that the four circles passing through the points

$$A'B'C,$$
$$AB'C',$$
$$A'BC',$$
$$ABC,$$

touch the antipolar of the Hart circle of a co-lunar triangle.

Thus this system of four circles touches the polar of the Hart circle of the original triangle, and the antipolars of the Hart circles of the three co-lunar triangles.

10. To find the radii of these four circles.

The radius r of the Hart circle of any spherical triangle is connected with the radius R of the circumscribing circle of that triangle by the relation

$$\tan R = 2 \tan r$$

(Salmon, *Solid Geometry*, Art. 261).

Hence the radius ρ of any of the four circles, to which the circles

$$ABC,$$
$$A'B'C,$$
$$A'BC',$$
$$AB'C',$$

are tangential, is given by the relation

$$2 \cot \rho = \cot r_1, \quad \text{or} \quad \tan \rho = 2 \tan r_1,$$

r_1 being the radius of the inscribed circle of one of the four spherical triangles ABC, $A'B'C$, $A'BC'$, $AB'C'$.

11. The sixty-four circles which can be drawn to touch three given circles 1, 2, 3 on a sphere and their three antipodal circles 1′, 2′, 3′, in such a manner that each touches 1 or 1′, 2 or 2′, and 3 or 3′, can be divided into two hundred and twenty-four groups of four, each of which touches another circle.

Eight circles can be drawn to touch each of the groups

$$123,\quad 1'2'3',$$
$$12'3',\quad 1'23,$$
$$1'23',\quad 12'3,$$
$$1'2'3,\quad 123'.$$

The theorem of Art. (7) shows that these sixty-four circles can be divided into $8 \times 14 = 112$ groups of four, each of which touches another circle.

Let us take, for considerations of symmetry in the distribution of the contacts, the case in which no two of the circles 1, 2, 3 intersect.

Let $a, b, c, d, a', b', c', d'$ denote the tangent circles of 1, 2, 3, the nature of their respective contacts with the latter being defined by the scheme

	a	b	c	d	a'	b'	c'	d'
1	*ext.*	*ext.*	*int.*	*int.*	*int.*	*int.*	*ext.*	*ext.*
2	*ext.*	*int.*	*ext.*	*int.*	*int.*	*ext.*	*int.*	*ext.*
3	*ext.*	*int.*	*int.*	*ext.*	*int.*	*ext.*	*ext.*	*int.*

and let

i, ii, iii, iv, v, vi, vii, viii denote their respective polars,

i′, ii′, iii′, iv′, v′, vi′, vii′, viii′, their respective antipolars.

Then a, b, c, d touch a Hart circle, all internally, in the case we have chosen; therefore (Lemmas α, β) i, ii, iii, iv touch the polar of this Hart circle, all internally.

It is also to be observed that i, ii, iii, iv do not form a Hart group in the polar figure.

For, let $\alpha, \beta, \gamma, \alpha', \beta', \gamma'$ be the respective polars and antipolars of 1, 2, 3; then, by the foregoing scheme, and Lemmas (α, β), the nature

of the contacts of i, ii, iii, iv with $\alpha, \beta, \gamma, \alpha', \beta', \gamma'$, is according to the scheme

	α	β	γ	α'	β'	γ'
i				*ext.*	*ext.*	*ext.*
ii		*int.*	*int.*	*ext.*		
iii	*int.*		*int.*		*ext.*	
iv	*int.*	*int.*				*ext.*

Thus i, ii, iii, iv touch a circle, although they do not touch any three of the circles $\alpha, \beta, \gamma, \alpha', \beta', \gamma'$.

Again, a, a', b, b' (a Hart group of the second species) touch a Hart circle in addition to 1, 2, 3. The nature of their contacts with 1, 2, 3 is defined by the first scheme; whence the nature of their contacts with their Hart circle is inferred according to the scheme

	a	b	a'	b'
1	*ext.*	*ext.*	*int.*	*int.*
2	*ext.*	*int.*	*int.*	*ext.*
3	*ext.*	*int.*	*int.*	*ext.*
Hart circle	*ext.*	*ext.*	*int.*	*int.*

Therefore the polar of this Hart circle touches i′, ii′ externally, and v, vi internally (Lemmas α, β).

It is further to be observed that i′, ii′, v, vi do not form a Hart

group in the polar figure, for the nature of their contacts with $\alpha, \beta, \gamma, \alpha', \beta', \gamma'$ is according to the scheme

	α	β	γ	α'	β'	γ'
i′	*ext.*	*ext.*	*ext.*			
ii′	*ext.*				*int.*	*int.*
v	*int.*	*int.*	*int.*			
vi	*int.*				*ext.*	*ext.*

Thus i′, ii′, v, vi touch a circle, although they do not touch any three of the circles $\alpha, \beta, \gamma, \alpha', \beta', \gamma'$.

Hence, corresponding to each of the fourteen Hart groups of the circles touching 1, 2, 3, we have a group of four circles touching α or α', β or β', and γ or γ', which touch a circle, although they are not tangential to the same three circles. Thus, the sixty-four circles which can be drawn to touch 1 or 1′, 2 or 2′, and 3 or 3′, can be divided into 8×14 Hart groups of four, each of which touches another circle; and also into 8×14 *other* groups of four, each of which touches another circle; *i.e.*, in all, into 224 groups of four, each of which touches another circle.

12. By joining the circles in the last Art. to the centre of the sphere by right circular cones, we observe that:—

The thirty-two right circular cones that can be drawn, with a given point as vertex, to touch three given right circular cones with that point as vertex, can be divided into one hundred and twelve groups of four, each of which is tangential to another right circular cone with that point as vertex.

13. In the same manner, from Art. (8), we deduce that:—

The thirty-two right circular cones that can be drawn, with a given point as vertex, to pass through the twelve lines of intersection, taken three by three, of three given right circular cones having th[redacted]

point as vertex, can be divided into thirty-two groups of four, each of which is tangential to four other right circular cones with that point as vertex.

This proposition may also be looked upon as a degenerate case of Art. (12), when the three given right circular cones become straight lines.

14. In the theorem of Art. (7), suppose the three given circles to be three great circles, and we deduce that:—

The four right circular cones that can be drawn, with a given point as vertex, to touch three given planes intersecting in that point, are tangential to four other right circular cones with that point as vertex.

This may likewise be looked upon as a degenerate case of the theorem of Art. (12).

15. In the theorem of Art. (8), suppose the three given circles to be three great circles, and we deduce that:—

The four right circular cones that can be drawn through three given concurrent straight lines touch four other right circular cones whose common vertex is the point of intersection of the three given straight lines.

This theorem may also be regarded as a degenerate case of the theorem of Art. (13).

16. The thirty-two conics that can be drawn having double contact with a given conic, and touching three given conics which have also double contact with this given conic, can be divided into one hundred and twelve groups of four, each of which touches another conic having double contact with the same conic.

In the theorem of Art. (12), project the sphere into a quadric surface; and the right circular cones become cones having their common vertex at the centre of the quadric, and having plane section with it.

Apply Lemma i., taking the centre of the quadric as origin, and any plane as plane of projection; the section of the figure by the plane of projection consists of three given conics, having double contact with the section of the asymptotic cone by the plane of projection, and the thirty-two conics that can be drawn touching them, and having double contact with the section of the asymptotic cone by the plane of projection; and the theorem enunciated follows at once from that of Art. (12).

17. The thirty-two conics which can be drawn having double contact with a given conic, and passing through three of the twelve points of intersection of three given conics which have double contact with this given conic, can be divided into thirty-two groups of four, each of which touches *four* conics having double contact with this given conic.

This follows from the theorem of Art. (13), by the method employed in the last Art.

18. The four conics that can be drawn touching the sides of a given triangle, and having double contact with a given conic, touch *four* conics which have double contact with this given conic.

This follows from the theorem of Art. (14), by the application of the same method.

19. The four conics that can be drawn through three given points, and having double contact with a given conic, touch *four* conics which have double contact with this given conic.

This follows from the theorem of Art. (15), by the application of the same method.

Thursday, January 14th, 1892.

Prof. GREENHILL, F.R.S., President, in the Chair.

Mr. Ralph Holmes, B.A., St. John's College, Cambridge, Mathematical Lecturer at King's College, London, and Mr. E. T. Dixon, Trinity College, Cambridge, were elected members.

On the motion of Mr. Walker, seconded by Mr. Brooksmith, the Treasurer's report was unanimously adopted.

The President drew the attention of the members present to the loss the Society had sustained by the death of Prof. Kronecker, who was elected an honorary member January 14th, 1875.

The following communications were made:—

The Harmonic Functions for the Elliptic Cone: Mr. Hobson.

Some Theorems relating to a System of Coaxal Circles: Mr. R. Lachlan.

Note on Dirichlet's Formula for the Number of Classes of Binary Quadratic Forms for a Complex Determinant: Prof. G. B. Mathews.

Researches in Calculus of Variations (third paper): Mr. E. P. Culverwell.

Impromptu Communications were made by Mr. Elliott and Major MacMahon.

The following presents were received :—

"Beiblätter zu den Annalen der Physik und Chemie," Vol. xv., No. 11; 1891.

"Atti del Reale Instituto Veneto di Scienze, Lettere, ed Arti," Vol. xxxviii., 7th series, Vol. i., Pt. 10; Vol. ii., Pts. 1–9.

"Physical Society of London—Proceedings," Vol. xi., Pt. 2, Dec., 1891.

"Nautical Almanac" for 1895.

"Proceedings of Royal Society," Vol. l., No. 303.

"Memoir on the Coefficients of Numbers," by B. Seal, M.A.; Calcutta, 1891.

"Bulletin de la Société Mathématique de France," Vol. xix., No. 7.

"Rendiconti del Circolo Matematico di Palermo," Vol. v., Pt. 6, 1891, Nov.–Dec.

"Archives Néerlandaises des Sciences Exactes et Naturelles," Vol. xxv., Pts. 3, 4; Harlem, 1891.

"Atti della Reale Accademia dei Lincei," Vol. vii., 9–11.

"Educational Times," January, 1892.

"Annals of Mathematics," Vol. vi., No. 3.

"Journal of College of Science, Japan," Vol. iv., Pt. 2; Tokyo, 1891.

"Engineering Review," Vol. i., No. 8, Dec. 5th, 1891.

"Acta Mathematica," xv., 3, 4.

"Transactions of Royal Irish Academy," Vol. xxix., Pt. 17.

"Cayley, Mathematical Papers," Vol. iv. (2 copies).

"Atti della R. Accademia di Napoli," 2nd Series, Vol. iv.; Napoli, 1891.

"Washington Observations, made during the year 1886, at the U. S. Naval Observatory;" Washington, 1891.

"Die Willkürlichen Functionen in der Mathematischen Physik—Inaugural-Dissertation zur Erlangung der Doctorwürde, zu Königsberg i. Pr." (24 Okt., 1891), von Arnold Sommerfeld; Königsberg in Pr., 1891.

"The Mathematical Magazine: a Journal of Elementary and Higher Mathematics," edited by A. Martin, Ph.D., Vol. ii., No. 5, Oct., 1891; Washington, 1891.

"A Direct and General Method of Finding the Approximate Values of the Real Roots of Numerical Equations to any Degree of Accuracy," by J. W. Nicholson, A.M.; New Orleans, 1891.

"The Life-Romance of an Algebraist," by George Winslow Pierce; Boston, 1891. From the Author.

Note on Dirichlet's Formula for the Number of Classes of Binary Quadratic Forms for a Complex Determinant. *By* Professor G. B. MATHEWS. Read January 14th, 1892. Received, in amended form, February 15th, 1892.

In the memoir on binary quadratic forms with complex coefficients (*Crelle*, XXIV., p. 291, or *Werke*, I., p. 612), Dirichlet has given a formula for the number of classes for a given determinant D. In the simplest case, when the determinant is not divisible by any square, and

$$D = \alpha + \beta i,$$

with $$\alpha \equiv 1 \pmod 4, \quad \beta \equiv 0 \pmod 2,$$

the formula in question reduces to

$$h = \frac{8\,|\,D\,|}{\pi \log \sigma} \cdot \Sigma \left(\frac{\lambda\alpha + \nu\beta}{m}\right) \frac{1}{\lambda^2 + \nu^2},$$

where $m = \alpha^2 + \beta^2$, $|D| = +\sqrt{\alpha^2 + \beta^2}$, $\left(\frac{\lambda\alpha + \nu\beta}{m}\right)$ is a Legendrian symbol of reciprocity, and λ, ν assume all real integral values, such that

$$\lambda \equiv 1 \pmod 4, \quad \nu \equiv 0 \pmod 2.$$

Moreover, σ denotes the norm of $(T + U\sqrt{D})$, where (T, U) is the fundamental solution of

$$T^2 - DU^2 = 1.$$

Dirichlet observes that the value of h may be expressed in finite terms by means of the elliptic functions of modulus $\sqrt{\frac{1}{2}}$, and then proceeds to say that although the result may be obtained without any difficulty, the appreciation of its true character depends upon the algebraical theory of the division of the period for these particular transcendents, the discussion of which he reserves for the second part of the memoir. This second part was never published, if indeed it was ever written; and I am not aware that anything has been done towards completing Dirichlet's investigation on the lines that he has indicated. In the following note, the explicit value of h, in terms of elliptic functions, is obtained by elementary methods for the particular case above given; there would be no particular difficulty in treating the other cases in a similar manner.

If r denotes $e^{2\pi i/m}$, then, by a formula due to Gauss (*Summatio quarumdam serierum*, &c.),

$$\left(\frac{\lambda a+\nu\beta}{m}\right)\sqrt{m}=\Sigma\left(\frac{k}{m}\right)r^{(\lambda a+\nu\beta)\kappa},$$

$$[\kappa=1, 2, 3 \ldots (m-1)],$$

with the convention that $\left(\frac{k}{m}\right)$ is to be put equal to zero, when k is not prime to m.

Hence $$h=\frac{8}{\pi\log\sigma}\Sigma\left(\frac{k}{m}\right)\frac{e^{2k\pi i(\lambda a+\nu\beta)/m}}{\lambda^2+\nu^2},$$

and this is easily transformed into

$$h=\frac{16}{\pi\log\sigma}\Sigma\left(\frac{k}{m}\right)\frac{\cos\frac{2\lambda ak\pi}{m}\cos\frac{2\nu\beta k\pi}{m}}{\lambda^2+\nu^2},$$

where λ now assumes all positive odd values, and ν all positive even values; and it is understood that when $\nu=0$, the corresponding term of the series is multiplied by $\frac{1}{2}$ instead of 1.

Now consider the elliptic functions for which

$$K'=K, \quad \kappa=\kappa'=\sqrt{\tfrac{1}{2}}, \quad q=e^{-\pi};$$

then (Jacobi, *Werke*, I., p. 155),

$$\log\mathrm{dn}\frac{2Kx}{\pi}=\log\sqrt{\kappa'}+\frac{2\cos 2x}{\sinh\pi}+\frac{2\cos 6x}{3\sinh 3\pi}+\ldots$$

Write successively

$$x=\frac{(a+\beta i)\pi}{m}, \qquad x=\frac{(a-\beta i)\pi}{m},$$

and add; thus $\log\left\{\mathrm{dn}\frac{2(a+\beta i)K}{m}\mathrm{dn}\frac{2(a-\beta i)K}{m}\right\}$

$$=\log\kappa'+\frac{4\cosh\frac{2\beta\pi}{m}}{\sinh\pi}\cos\frac{2a\pi}{m}+\frac{4\cosh\frac{6\beta\pi}{m}}{3\sinh 3\pi}\cos\frac{6a\pi}{m}+\ldots$$

Changing β into $\frac{m}{2}-\beta$, $=\beta'$ say, and making use of the formula

$$\mathrm{dn}(u-iK')=-\frac{i\,\mathrm{cn}\,u}{\mathrm{sn}\,u},$$

we obtain

$$\text{constant}+\log\left\{\frac{\operatorname{cn}\frac{2(a+\beta i)K}{m}\operatorname{cn}\frac{2(a-\beta i)K}{m}}{\operatorname{sn}\frac{2(a+\beta i)K}{m}\operatorname{sn}\frac{2(a-\beta i)K}{m}}\right\}$$

$$=\log\kappa'+\frac{4\cosh\frac{2\beta'\pi}{m}}{\sinh\pi}\cos\frac{2a\pi}{m}+\frac{4\cosh\frac{6\beta'\pi}{m}}{3\sinh 3\pi}\cos\frac{6a\pi}{m}+\ldots.$$

Adding this to the previous formula, and writing, for convenience,

$$\psi(u)=\frac{\operatorname{cn}u\operatorname{dn}u}{\operatorname{sn}u},$$

we have

$$\log\left\{\psi\left(\frac{\overline{a+\beta i}.K}{m}\right)\psi\left(\frac{\overline{a-\beta i}.K}{m}\right)\right\}$$

$$=C+\frac{4\left\{\cosh\frac{2\beta\pi}{m}+\cosh\frac{2\beta'\pi}{m}\right\}}{\sinh\pi}\cos\frac{2a\pi}{m}$$

$$+\frac{4\left\{\cosh\frac{6\beta\pi}{m}+\cosh\frac{6\beta'\pi}{m}\right\}}{3\sinh 3\pi}\cos\frac{6a\pi}{m}+\ldots,$$

where C is a constant.

Now (Todhunter's *Int. Calc.*, p. 300, 4th edition),

$$\frac{\cosh a(\pi-x)+\cosh ax}{a\sinh a\pi}=\frac{2}{\pi}\left\{\frac{1}{a^2}+\frac{2\cos 2x}{2^2+a^2}+\frac{2\cos 4x}{4^2+a^2}+\ldots\right\};$$

consequently

$$\log\left\{\psi\left(\frac{\overline{a+\beta i}.K}{m}\right)\psi\left(\frac{\overline{a-\beta i}.K}{m}\right)\right\}$$

$$=C+\frac{8}{\pi}\left\{\frac{1}{1^2}+\frac{2\cos\frac{4\beta\pi}{m}}{2^2+1^2}+\frac{2\cos\frac{8\beta\pi}{m}}{4^2+1^2}+\ldots\right\}\cos\frac{2a\pi}{m}$$

$$+\frac{8}{\pi}\left\{\frac{1}{3^2}+\frac{2\cos\frac{12\beta\pi}{m}}{2^2+3^2}+\frac{2\cos\frac{24\beta\pi}{m}}{4^2+3^2}+\ldots\right\}\cos\frac{6a\pi}{m}$$

$$+\ldots\ldots.$$

Comparing this with the expression for h, and observing that, for a complete set of residues, $\Sigma\left(\frac{k}{m}\right)=0$, so that C need not be determined, we find that

$$h\log\sigma=\Sigma\left(\frac{k}{m}\right)\log\left\{\psi\left(\frac{k\,(a+\beta i)\,K}{m}\right)\psi\left(\frac{k\,(a-\beta i)\,K}{m}\right)\right\},$$

$$\left[k=1, 2, 3\ldots(m-1)\right].$$

If a stands for a quadratic residue of m, and b for a non-residue, we may write

$$h\log\sigma=\log\frac{\Pi\psi\left(\frac{a\,(a+\beta i)\,K}{m}\right)\psi\left(\frac{a\,(a-\beta i)\,K}{m}\right)}{\Pi\psi\left(\frac{b\,(a+\beta i)\,K}{m}\right)\psi\left(\frac{b\,(a-\beta i)\,K}{m}\right)}.$$

It is interesting to see that we thus obtain, in an elliptic function form, an expression for

$$Nm\,(T_h+U_h\sqrt{D}),$$

where (T_h, U_h) is a solution of $T^2-DU^2=1$.

Objections might no doubt be raised to the details of the preceding analysis; in fact, it cannot be said to be rigorous; still I believe the result is correct, and that the process may be justified to some extent by observing that the transcendental series equated to each other present exactly the same kind of discontinuities at the same critical points.

It ought to be said that the series for h in its first form is similar to those considered by Kronecker (Berlin *Sitzungsber.*, July, 1885), so that the result finally obtained ought to agree with his results (*l.c.*, p. 764); but, on account of the difference of notation, and the fact that Dirichlet's λ, ν do not agree with Kronecker's m, n, the identification is very troublesome.

Anyone who is inclined to undertake a numerical verification might consult with advantage Schwering's memoir on the "Multiplication of the Lemniscate Function" (*Crelle*, CVII., p. 196); it should be observed, however, that Schwering's $\mathrm{sn}\,u$ does not correspond to the $\mathrm{sn}\,u$ of the present note, but is Gauss's sinlemn u.

Thursday, February 11th, 1892.

Professor GREENHILL, F.R.S., President, in the Chair.

Messrs. E. T. Dixon and R. Holmes were admitted into the Society. Mr. James William Nicholson, M.A., President and Professor of Mathematics, Louisiana State University, U.S.A., was elected a member.

The following communications were made:—

On the Logical Foundations of Applied Mathematical Science: Mr. Dixon.

Note on the inadmissibility of the usual reasoning by which it appears that the Limiting Value of the Ratio of two Infinite Functions is the same as the ratio of their first derived, with instances in which the result obtained by it is erroneous: Mr. Culverwell.

On Saint-Venant's Torsion of Prisms: Mr. Basset.

The following presents were received:—

"Proceedings of the Mathematical Society of Edinburgh," Vols. II., III., IV., V., VI., VIII., IX.

"Beiblätter zu den Annalen der Physik und Chemie," Band XV., St. 12, 1891 Band XVI., St. 1, 1892.

"Royal Society—Proceedings," Vol. L., No. 304, January, 1892.

"Nyt Tidsskrift for Mathematik," B. Anden Aargang, Nos. 2, 3, 4.

"Leipzig. Gesells. der Wissenschaften — Berichte über die Verhandlungen," Math. Phys. Classe, 1891, III.

"Bulletin des Sciences Mathématiques," 2e Série, Tome XV., December, 1891.

"Leipzig. Gesells. der Wissenschaften — Ueber einen eigenthümlichen Fall Elektrodynamischer Induction," II.

"Educational Times," February, 1892.

"Atti della R. Accad. dei Lincei—Rendiconti," Vol. VII., Fasc. 12, e Indice del Volume, 2° Semestre, 1891; Roma.

"Cambridge Philosophical Society—Proceedings," Vol. VII., Pt. 5.; "Transactions," Vol. XV., Pt. 2.

"Toulouse—Annales de la Faculté des Sciences," Tome V., 1891, Fasc. 3, 4.

"Napoli—Rendiconto dell' Accademia delle Scienze Fisiche e Matematiche," Serie 2, Vol. V., Fasc. 1–12, 1891.

"Surveyor," Vol. I., Nos. 1, 2, 3, 1892.

"Œuvres Complètes de Christiaan Huygens," Vol. IV.; Le Haye, 1891.

"Royal Society's Catalogue of Scientific Papers," 1874–1883, Vol. IX.

"Mendizábul Tamborrel (J. de), Tables des Logarithmes," Paris, 1891.

Thursday, March 10th, 1892.

Professor GREENHILL, F.R.S., President, in the Chair.

Abinas Chandra Basu, Professor of Mathematics, Agra College, Agra, was elected a member.

The President and Mr. S. Roberts spoke of the services rendered by the late Dr. Hirst to the Society, and upon the loss sustained by the mathematical world in consequence of his decease.

The President announced that the Council had just selected the six following mathematicians for the compliment of foreign membership—viz., Messrs. Poincaré, Hertz, Mittag-Leffler, Schwarz, Beltrami, and Willard Gibbs; and, in accordance with Rule 29, he nominated them for election at this the first subsequent ordinary meeting, with a view to their being ballotted for at the April meeting.

The following communications were made:—

The simplest Equivalent of a given Optical Path, and the observations required to determine it: Dr. J. Larmor.

On cases in which a Hyperelliptic Integral of the First Order can be expressed as the sum of two Elliptic Integrals: Professor W. Burnside.

On the Analytical Theory of the Congruency: Professor Cayley.

Notes on Dualistic Differential Transformations: Mr. E. B. Elliott.

On certain Curves of the Fourth Order and the Porism of the Inscribed and Circumscribed Polygon: Mr. R. A. Roberts.

Professor M. J. M. Hill made a few remarks on Singular Solutions, and the President spoke on the Rectification of the Cartesian Oval.

A Cabinet Likeness of Mr. S. Roberts was presented by that gentleman to the Album.

The following presents were received:—

"Beiblätter zu den Annalen der Physik und Chemie," Band XVI., St. 2; Leipzig, 1892.

"Actuaries—Journal of Institute of," Vol. XXIX., Pt. 6, January, 1892.

"Jahrbuch über die Fortschritte der Mathematik," Jahrgang 1889, Band XXI., Heft 1; Berlin, 1892.

"Nieuw Archiev voor Wiskunde," Deel XIX., St. 1–2; Amsterdam, 1892.

"Jornal de Sciencias Mathematicas e Astronomicas," Vol. x., No. 4; Coimbra, 1892.

"Bulletin des Sciences Mathématiques," 2nd Series, Tome XVI., January, 1892.

"Sitzungsberichte der K. Preuss. Akademie der Wissenschaften zu Berlin," 41–53, and Jahrgang 1891; Berlin, 1891.

"Atti della Reale Accademia dei Lincei—Rendiconti," Vol. I., Fasc. 1–2, Sem. 1; Roma, 1892.

"Educational Times," March, 1892.

"Annals of Mathematics," Vol. VI., No. 4; University of Virginia, January, 1892.

"Journal für die reine und angewandte Mathematik," Band CIX, H. 2.

"Bollettino delle Pubblicazione Italiane ricevute per diritto di stampa," 1892, No. 148, in duplicate; Firenze.

The Simplest Specification of a given Optical Path, and the Observations required to determine it. By J. LARMOR. Read March 10th, 1892.

1. The complete specification of an optical path through a heterogeneous medium like the atmosphere, or through a combination of transparent substances which may form an optical instrument, requires not merely the form of the curve which a narrow beam or filament of light traverses, but also a statement of the character of the modification impressed upon the filament by traversing that path. The mode of division of a beam of light into filaments, or linear elements of waves, in the way thus suggested, corresponds much more closely to the physical reality than the more ordinary analysis which splits these filaments up into the rays of which they are supposed to be constituted; and it is fortunate that, by making use of the dioptrical methods introduced by Sir W. R. Hamilton, and further developed more recently by Maxwell, the consideration of filaments leads to a deeper and more coordinated analysis than that of rays. That this is the case follows from the fact that the effects impressed upon all filaments by traversing a given path may be determined in a simple manner by observation or calculation relating to a few cases, and may be considered as expressed in terms of certain optical properties of the path. In addition to the geometrical gain which results from recognising that the treatment of a group of rays is nearly as simple as that of a single ray, there is the fact that all

phenomena of diffraction, arising from limitation of the beam, are most simply connected with the form of the wave-fronts which determine the filaments.

2. In the *Proceedings*, Vol. xx. (1889), p. 185, it has been explained, after Maxwell,* that the characteristic function for an optical path, from a point $(x_1, y_1, 0)$ at one end of it to a point $(x_2, y_2, 0)$ at the other end, is of the form

$$\begin{aligned} U = \text{const.} &+ \tfrac{1}{2}a_1x_1^2 + c_1x_1y_1 + \tfrac{1}{2}b_1y_1^2 \\ &+ px_1x_2 + qx_1y_2 + rx_2y_1 + sy_1y_2 \\ &+ \tfrac{1}{2}a_2x_2^2 + c_2x_2y_2 + \tfrac{1}{2}b_2y_2^2, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1) \end{aligned}$$

the terms of the first degree being omitted when the axes of z_1 and z_2 are tangential to the path.

The course of a ray, from a point $(\xi_1, \eta_1, \gamma_1^{-1})$ at one end to a point $(\xi_2, \eta_2, \gamma_2^{-1})$ at the other end, was then determined by the fact that the ray must strike the planes $z_1 = 0$ and $z_2 = 0$ at points $(x_1, y_1, 0)$ and $(x_2, y_2, 0)$, which satisfy the condition of making the characteristic function stationary for all variations of these points. When the media beyond the two ends of the path are homogeneous and isotropic, this leads to the determination of these points by the equations

$$\left.\begin{aligned} (\mu_1\gamma_1 + a_1)\,x_1 + c_1y_1 + px_2 + qy_2 &= \mu_1\gamma_1\xi_1 \\ c_1x_1 + (\mu_1\gamma_1 + b_1)\,y_1 + rx_2 + sy_2 &= \mu_1\gamma_1\eta_1 \\ px_1 + ry_1 + (\mu_2\gamma_2 + a_2)\,x_2 + c_2y_2 &= \mu_2\gamma_2\xi_2 \\ qx_1 + sy_1 + c_2x_2 + (\mu_2\gamma_2 + b_2)\,y_2 &= \mu_2\gamma_2\eta_2 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (2)$$

When, however, the media beyond the two ends of the path are isotropic but not homogeneous, these equations determine the inclinations of the ray, as it enters the path at the point $(x_1, y_1, 0)$, and emerges from it at the point $(x_2, y_2, 0)$; for its direction cosines

$$(l_1, m_1, n_1) \text{ and } (l_2, m_2, n_2)$$

at these two points are equal to

$$\{\gamma_1(\xi_1 - x_1),\ \gamma_1(\eta_1 - y_1),\ 1\} \text{ and } \{\gamma_2(\xi_2 - x_2),\ \gamma_2(\eta_2 - y_2),\ 1\},$$

* *Proc. Math. Soc.*, vi. (1874); *Collected Papers*, ii., p. 381.

which are

$$\left\{\frac{1}{\mu_1}\frac{\partial U}{\partial x_1}, \frac{1}{\mu_1}\frac{\partial U}{\partial y_1}, 1\right\} \text{ and } \left\{\frac{1}{\mu_2}\frac{\partial U}{\partial x_2}, \frac{1}{\mu_2}\frac{\partial U}{\partial y_2}, 1\right\}, \quad \ldots\ldots\ldots (3)$$

as they ought to be from the fundamental properties of the characteristic function. This remark in fact suggests an alternative method of stating the proof of the formulæ (2).

When the determinantal equation

$$\Delta \equiv \begin{vmatrix} \mu_1\gamma_1+a_1 & c_1 & p & q \\ c_1 & \mu_1\gamma_1+b_1 & r & s \\ p & r & \mu_2\gamma_2+a_2 & c_2 \\ q & s & c_2 & \mu_2\gamma_2+b_2 \end{vmatrix} = 0 \quad \ldots\ldots\ldots (4)$$

is satisfied, there is a plane sheaf of rays from the point $(\xi_1, \eta_1, \gamma_1^{-1})$ to the point $(\xi_2, \eta_2, \gamma_2^{-1})$, which intersect the plane $z_1 = 0$ along the line (of constant direction)

$$Sx_1 + Qy_1 = \text{constant}, \quad \ldots\ldots\ldots\ldots\ldots\ldots (5_1)$$

and also the plane $z_2 = 0$ along the line

$$Qx_2 + Py_2 = \text{constant}; \quad \ldots\ldots\ldots\ldots\ldots\ldots (5_2)$$

while we have the identical relation

$$A_1\mu_1\gamma_1\xi_1 + C_1\mu_1\gamma_1\eta_1 + P\mu_2\gamma_2\xi_2 + Q\mu_2\gamma_2\eta_2 = 0, \quad \ldots\ldots\ldots\ldots\ldots (6)$$

the capital letters representing the minors of the corresponding small-letter constituents in the determinant Δ.

The interpretation of these formulæ is that, under the condition (4), a plane sheaf of rays, instead of a single one, passes from any point in the plane $z_1 = \gamma_1^{-1}$ to any point in the plane $z_2 = \gamma_2^{-1}$; that the planes of all these sheaves are parallel at each end of the path, being given by the equations (5_1) and (5_2); that all rays issuing from a point in the plane $z_1 = \gamma_1^{-1}$ converge to a line in the plane $z_2 = \gamma_2^{-1}$; or, more generally, all rays issuing from the line

$$A_1\mu_1\gamma_1\xi_1 + C\mu_1\gamma_1\eta_1 = H \quad \ldots\ldots\ldots\ldots\ldots (7_1)$$

converge to the line

$$P\mu_2\gamma_2\xi_2 + Q\mu_2\gamma_2\eta_2 = -H, \quad \ldots\ldots\ldots\ldots\ldots (7_2)$$

and *vice versâ*.

3. The equation (4) establishes a 2-to-2 correspondence between the planes $z_1 = \gamma_1^{-1}$ and $z_2 = \gamma_2^{-1}$. Any point-focus on one side of the path has, for its conjugate, two focal lines on the other side of the path, whose distances from the origin are determined by this equation, and whose actual positions are determined by the equations (7).

The roots of (4), treated as an equation in γ_1 or γ_2, are necessarily always real; a result which admits of extension to any determinantal equation of this type, and so leads to a generalization of the proposition which asserts the reality of the roots of the period equation for the vibrations of a frictionless dynamical system with a given energy function. The roots of any determinantal equation of this type are in fact separated by those of the equation obtained by equating to zero the leading minor determinant; and in this way a system of equations of orders diminishing by units may be obtained, each of which separates the roots of the one before it, thus showing, on Sturm's principles, that the roots of each are all real.

When the two values of γ_1 in (4) are equal, their common value might at first sight be taken to be the root of either first minor; this would lead to

$$-\begin{vmatrix} \mu_2\gamma_2+a_2 & c_2 \\ c_2 & \mu_2\gamma_2+b_2 \end{vmatrix} \mu_1\gamma_1 = \begin{vmatrix} a_1 & p & q \\ p & \mu_2\gamma_2+a_2 & c_2 \\ q & c_2 & \mu_2\gamma_2+b_2 \end{vmatrix} = \begin{vmatrix} b_1 & r & s \\ r & \mu_2\gamma_2+a_2 & c_2 \\ s & c_2 & \mu_2\gamma_2+b_2 \end{vmatrix},$$

so that there would be two values of γ_2 which lead to equal roots for γ_1. But this argument is vitiated by the fact (noted below) that the values of γ_2 thus obtained are imaginary, while Sturm's theory relates only to equations with real coefficients. A direct procedure shows that there are four such values of γ_2.

The interpretation of these results requires some care. It might appear at first sight that any focus in the plane $z_2 = \gamma_2^{-1}$ has two coincident focal lines, and therefore a conjugate focus, in the plane $z_1 = \gamma_1^{-1}$; and thus that there are always four pairs of planes of conjugate foci. But we shall see presently, on other grounds, that this cannot generally be true. The explanation of the apparent paradox is that the equation (4) only insures that a certain plane sheaf of rays from the one point converges as a plane sheaf to the other point; and that the equality of the roots further insures that two plane sheaves from the one point converge as plane sheaves to the other point. But any two plane sheaves from a point to another do not make up a wave-front; that they should do so another condition is required, viz.,

that the time of passage should be the same for each—in other words, that they should have the same characteristic function; and this has not been secured.

The state of matters may be illustrated geometrically by the Poncelet 2-to-2 correspondence of points on two conics. Each point on a conic is given by its parameter; to a point on one, say the outer, there correspond, on the inner, the two points of contact of tangents from it; to a point on the inner, there correspond, on the outer, the two intersections of the tangent at it. There are four points of intersection of two conics, and four common tangents, which are the correlatives of the two sets of four points referred to in the last paragraph. It is at once evident geometrically that the correspondence is not symmetrical; where a coincidence occurs on one conic it does not occur on the other. To make a coincidence at the same point on both conics, the conics must touch at that point, instead of intersecting. Similarly, to have conjugate foci, not only must the values of γ_1 obtained from (4) be equal, but also the values of γ_2; to make the conditions for both these equalities consistent, a relation must hold between the coefficients, and therefore the optical path must have a special character.

That the geometrical may agree with the optical correspondence, in referring only to real points, the conic which we have called the inner one must lie wholly inside the other; otherwise the tangents will become imaginary over a certain range of the correspondence. The conics cannot therefore cross each other in real points. Thus the values of γ_2 which give equal roots for γ_1 in (4) must be imaginary, a fact which we have already used in the analytical theory.

3. If we revert for a moment to the more general question in which no part of the medium is homogeneous, the planes $z_1 = 0$, $z_2 = 0$ will be in correspondence if

$$\Delta' \equiv \begin{vmatrix} a_1 & c_1 & p & q \\ c_1 & b_1 & r & s \\ p & r & a_2 & c_2 \\ q & s & c_2 & b_2 \end{vmatrix} = 0. \quad \ldots\ldots\ldots\ldots\ldots\ldots (4')$$

The rays from a point in the plane $z_1 = 0$, which emerge from the plane $z_2 = 0$ in a constant direction, there form a plane sheaf parallel to the plane

$$S'x_1 + Q'y_1 = 0, \quad \ldots\ldots\ldots\ldots\ldots\ldots (5')$$

and *vice versâ*. The identical relation

$$A'_1\mu_1 l_1 + C'_1\mu_1 m_1 + P'\mu_2 l_2 + Q'\mu_2 m_2 = 0 \quad \ldots\ldots\ldots\ldots(6')$$

connects the direction cosines $(l_1, m_1, 1)$, $(l_2, m_2, 1)$ of the same ray at incidence and emergence. In this statement, the capital letters represent the minors of the determinant Δ'.

4. These considerations will in their analytical form be much simplified if we refer the path to its two cardinal points, so that in the characteristic function $p = s$, while q and r vanish. The determination of the emergent beam

$$V_2 = \mu_2 z_2 + \tfrac{1}{2}A_2 x_2^2 + C_2 x_2 y_2 + \tfrac{1}{2}B_2 y_2^2,$$

which corresponds to an incident beam

$$V_1 = \mu_1 z_1 + \tfrac{1}{2}A_1 x_1^2 + C_1 x_1 y_1 + \tfrac{1}{2}B_1 y_1^2,$$

is completed in § 15 of the previous paper already quoted (*Proceedings*, xx., p. 190). The result is

$$\left.\begin{aligned} &\frac{a_2+A_2}{b_1+A_1} = \frac{b_2+B_2}{a_1+B_1} = -\frac{c_2+C_2}{c_1+C_1} = \frac{p^2}{\Delta_1}, \\ \text{where} \quad &\Delta_1 = (a_1+A_1)(b_1+B_1) - (c_1+C_1)^2. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(8)$$

To obtain a pair of conjugate foci, we must have together

$$A_1 = B_1, \quad C_1 = 0,$$

$$A_2 = B_2, \quad C_2 = 0,$$

which leads, on subtraction of numerators and of denominators in (8), to a relation between the constants

$$\ldots \qquad \frac{a_2-b_2}{a_1-b_1} = \frac{c_2}{c_1}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(9)$$

This is exactly the condition that, by a rotation of both systems of axes through the same angle, the characteristic may assume the form

$$U = \text{const.} + \tfrac{1}{2}a_1 x_1^2 + \tfrac{1}{2}b_1 y_1^2 + p\,(x_1 x_2 + y_1 y_2) + \tfrac{1}{2}a_2 x_2^2 + \tfrac{1}{2}b_2 y_2^2. \quad \ldots(10)$$

The path is then such that the equations connecting x_1, x_2 are independent of the equations connecting y_1, y_2, each set being the same as those for an ordinary two-dimensional Gaussian system, except as

regards an absolute rotation round the tangent to the path at either end. The focal lines corresponding to all foci will then be parallel, and will determine the directions of these principal sections at the two ends of the path.

To pass to the case of symmetry round the axis, two further conditions are necessary, $a_1 = b_1$, and $a_2 = b_2$.

5. It will be observed that there are these two pairs of planes of conjugate foci, and no more; the conics of the analogy have double contact, and cannot touch again. The images formed in these planes will have a linear correspondence, and by means of them the path of any ray may be constructed. For the ray which connects any two points on one side must connect their image points on the other side, when such exist.

The fact that it is only under a special condition that two sets of image points exist, and that there cannot be more than two sets, may be more directly demonstrated, as in the following sketch. Let the points (ax, by, z) and $(ax, by, z_1 + R_1)$ on one side correspond to the points (x, y, z_2) and $(ax\cos\epsilon - \beta y \sin\epsilon,\ ax\sin\epsilon + \beta y\cos\epsilon,\ z_2 + R_2)$ on the other side, the correspondence of the second pair thus involving a rotation through an angle ϵ. As the incident rays form a parallel system, the emergent rays must be normal to a surface

$$z = z_2 + \tfrac{1}{2}Ax^2 + Cxy + \tfrac{1}{2}By^2.$$

Therefore, for all values of x, y,

$$\frac{(1-a\cos\epsilon)\,x + \beta\sin\epsilon\, y}{Ax + Cy} = \frac{-a\sin\epsilon\, x + (1-\beta\cos\epsilon)\,y}{Cx + By} = \frac{-R_2}{1};$$

therefore
$$\left.\begin{aligned} 1 - a\cos\epsilon &= -R_2 A \\ \beta\sin\epsilon &= -R_2 C \end{aligned}\right\}, \qquad \left.\begin{aligned} a\sin\epsilon &= R_2 C \\ 1 - \beta\cos\epsilon &= -R_2 B \end{aligned}\right\},$$

so that $\epsilon = 0$, or else $a + \beta = 0$.

The first alternative gives principal sections at each end of the beam, the case discussed above. The second alternative requires also, by consideration of a parallel beam incident on the opposite side of the path, that $a + b = 0$; thus the two image planes must be similar to the two object planes, but turned through different angles; but even then it will be found that any other beam incident normally to one surface will not emerge normally to another.

6. The theorem of *Proceedings*, xx., p. 192, which gives the simplest equivalent of a general optical path, may be expressed in a somewhat simpler form, which is worth notice. The modification follows on observing that an optical instrument symmetrical round its axis may be replaced by two thin lenses at any two specified points on its axis.

Consider any unsymmetrical optical system, separating two homogeneous media; and let the path of a central ray through it be considered as its axis. Then the system is optically equivalent to two thin astigmatic lenses placed at definite points O_1 and O_2 on a straight axis; and that in two ways. It may, however, be necessary, in order to make these points real, to add on a simple lens to the original system. The positions of these cardinal points may be found experimentally, without any previous knowledge of the constitution of the instrument, on making the incident pencil symmetrical round the axis by the use of a circular stop, and shifting the luminous point from which it comes, until, on emergence into the second medium, it diverges symmetrically from either of its focal lines, as may be tested by focussing that line in an observing telescope and examining the appearance when it is put slightly out of focus. The focus of the incident beam is then the point O_1, and the circular section of the emergent beam is the point O_2. The determination may be checked by passing a beam over the path in the opposite direction. The astigmatic lens to be placed at O_2 is now determined as that lens which will refract an incident pencil from O_1 to the same focal lines as the actual instrument does; these focal lines are parallel to the principal sections of the lens, and its focal lengths in these sections are at once determined. In a similar manner the astigmatic lens to be placed at O_1 is determined. The instrument is then optically equivalent in all respects to this pair of lenses, with the exception that all emergent beams may in addition have to be rotated round the axis through a constant angle, which, however, is optically of no consequence. When the axis of the emergent beam is in a different direction from that of the incident beam, these points O_1, O_2 must be laid off on a subsidiary straight axis. The distance between them is to be adjusted by the fact that the section on the plane O_2 of a given beam from the side of O_1 must be the same on the subsidiary axis as on the actual path. The astigmatic lenses are then determined, with reference to the distance O_1O_2, and in the result the incident and emergent beams will correspond exactly in both systems.

The system of observations here sketched therefore suffices

theoretically to determine the optical effect of any transparent system, however complicated, on all beams of light passing across it in a given path, straight or curved.

On the Form of Hyperelliptic Integrals of the First Order, which are expressible as the Sum of two Elliptic Integrals. By W. BURNSIDE. Received March 7th, 1892. Read March 10th, 1892.

The reduction of the integral

$$\int \frac{(a+bx)\,dx}{\sqrt{x\,(x-1)(x-\kappa)(x-\lambda)(x-\kappa\lambda)}}$$

to the sum of two elliptic integrals is due originally to Jacobi.

The transformation is most simply exhibited in the following form.

If
$$z = x + \frac{\kappa\lambda}{x},$$

then
$$(x-1)(x-\kappa\lambda) = x\,(z-1-\kappa\lambda),$$
$$(x-\kappa)(x-\lambda) = x\,(z-\kappa-\lambda),$$
$$x-\sqrt{\kappa\lambda} = \sqrt{x}\,\sqrt{z-2\sqrt{\kappa\lambda}},$$
$$x+\sqrt{\kappa\lambda} = \sqrt{x}\,\sqrt{z+2\sqrt{\kappa\lambda}},$$
$$(x-\sqrt{\kappa\lambda})(x+\sqrt{\kappa\lambda})\,dx = x^2 dz,$$

and hence

$$\frac{(x+\sqrt{\kappa\lambda})\,dx}{\sqrt{x\,(x-1)(x-\kappa)(x-\lambda)(x-\kappa\lambda)}} = \frac{dz}{\sqrt{(z-2\sqrt{\kappa\lambda})(z-1-\kappa\lambda)(z-\kappa-\lambda)}},$$

and

$$\frac{(x-\sqrt{\kappa\lambda})\,dx}{\sqrt{x\,(x-1)(x-\kappa)(\kappa-\lambda)(x-\kappa\lambda)}} = \frac{dz}{\sqrt{(z+2\sqrt{\kappa\lambda})(z-1-\kappa\lambda)(z-\kappa-\lambda)}}.$$

It has also been shown by Herr Königsberger (*Borchardt's*

Journal, Vol. LXVII.) that if a hyperelliptic integral of the first order is reducible to the sum of two elliptic integrals by a quadratic substitution, it can always, by a linear substitution, be expressed in the above form.

I discuss in this paper the corresponding question for rational substitutions of a higher order than the second. For the case of a cubic transformation the problem is treated completely, the general form of the integral, and the two substitutions that are necessary to transform it, being given explicitly. For the case of the quartic and quintic transformations the equations of transformation are explicitly obtained with the general forms of the integrals to which they apply.

The general theory of hyperelliptic integrals of the first order shows that when one integral of the first species belonging to the equation

$$y^2 = f_6(x),$$

a sextic function, can be found with only two independent periods, a second with the same property always exists; and that, if suitable multiples of these are chosen as the normal integrals, the four periods of each are expressible in the form

$$w_1 \ldots\ldots 1, \quad 0, \quad \omega, \quad \frac{1}{n},$$

$$w_2 \ldots\ldots 0, \quad 1, \quad \frac{1}{n}, \quad \omega',$$

where n is an integer.

It follows from this that a double θ-function, whose arguments are any two independent integrals of the first species, can in this case, by a transformation of the n^{th} order, be expressed as a product of two simple θ-functions, and hence that the normal integrals can, by transformations of the n^{th} order, be reduced to elliptic integrals. (See a paper by the late Mme. S. Kowalevski, *Acta Mathematica*, Vol. IV.) The transformations will generally be distinct; that they are the same for the above-quoted case of the quadratic substitution results from the fact that for that case the equation

$$s^2 = f_6(x)$$

is rationally transformable into itself.

These considerations suggest starting with the elliptic integral, and determining the form of the substitution which will transform it

into a hyperelliptic integral of the first order. If the elliptic integral be of the first species, *i.e.*, everywhere finite, the hyperelliptic integral will necessarily be so also.

Suppose, then, that the elliptic differential

$$\frac{dz}{\sqrt{z-\alpha \,.\, z-\beta \,.\, z-\gamma \,.\, z-\delta}}$$

is transformed by the substitution of the n^{th} order

$$z = \frac{U}{V},$$

where U, V are functions of x, either both of the n^{th} degree, or one of the n^{th} and one of the $n-1^{\text{th}}$; and for the present let n be odd. If the result is a hyperelliptic differential of the first order,

$$(U-\alpha V)(U-\beta V)(U-\gamma V)(U-\delta V)$$

must be the product of a sextic function of x, and a perfect square; and this can only be the case when three of the factors are products of linear functions, and perfect squares, while the fourth is the product of a cubic function and a square.

The substitution may therefore be written generally in the form

$$z-\alpha : z-\beta : z-\gamma : z-\delta$$

$$:: (x-x_1)\, X_1^2 : (x-x_2)\, X_2^2 : (x-x_3)\, X_3^2 : (x-x_4)(x-x_5)(x-x_6)\, X_4^2,$$

where X_1, X_2, X_3 are rational integral functions of x of degree $\frac{n-1}{2}$, and X_4 is of degree $\frac{n-3}{2}$.

By linear substitutions, performed on z and x, this may always be written in the standard form

$$z : z-1 : 1 : z-z_0$$

$$:: xX_1^2 : (x-1)\, X_2^2 : X_3^2 : (x-\kappa)(x-\lambda)(x-\mu)\, X_4^2,$$

the coefficients of the leading terms in X_1, X_2, X_4 being unity (or all the same).

The relations between the constants in this equation are mos easily obtained by a consideration of the corresponding Riemann's surface.

The surface is n-sheeted and simply connected.

At each of the points $z = 0, 1, \infty$, $n-1$ of the sheets are connected

in pairs by branch-points, and at $z = z_0$, $n-3$ sheets are so connected in pairs; this gives $2n-3$ branch-points, so that there must be one more to make the surface simply connected. Hence the equation of transformation can be written more completely

$$z : z-1 : 1 : z-z_0 : z-z_1$$

$$:: xX_1^2 : (x-1)X_2^2 : X_3^2 : (x-\kappa)(x-\lambda)(x-\mu)X_4^2 : X_5(x-x_1)^2,$$

where X_5 is a rational integral function with x^{n-2} for its leading term.

Now when z_0 and z_1 are given, there is only a finite number of ways of connecting the sheets of the surface together, and, for each way of connecting, x is a perfectly definite function of z, since it is a one-valued function on the Riemann's surface, with given values at three given points. Hence the constants in the above equation are algebraic functions of z_0 and z_1; or, more conveniently, of two arbitrary parameters.

When κ, λ, μ are so determined as functions of two arbitrary parameters,

$$\int \frac{(a+bx)\,dx}{\sqrt{x(x-1)(x-\kappa)(x-\lambda)(x-\mu)}}$$

is a hyperelliptic integral of the first order, which, by transformations of the n^{th} order, can be expressed as the sum of two elliptic integrals, and the method by which it has been obtained shows that it can only differ by a linear transformation from the most general integral so expressible. The above transformation only changes the hyperelliptic integral into an elliptic integral for a particular value of the ratio a/b; the general theory shows that another transformation of the n^{th} order must exist, which will apply to another value of the ratio.

The differential coefficient dz/dx vanishes only at the finite-branch points of the surface; hence

$$\frac{dz}{dx} = \frac{X_1 X_2 X_4 (x-x_1)}{X_3^2} \times \text{constant};$$

and therefore

$$\frac{dz}{\sqrt{z(z-1)(z-z_0)}} = \frac{\text{constant} \times (x-x_1)\,dx}{\sqrt{x(x-1)(x-\kappa)(x-\lambda)(x-\mu)}},$$

determining the particular value of the ratio a/b to which the given substitution applies.

Cubic Transformation.

The equation for the cubic transformation involves only one arbitrary parameter, and so differs from the general case of any odd number; for in this case there can only be four branch-points on the surface, and of these three can be chosen arbitrarily. The coefficients in the hyperelliptic integral will, however, still involve two arbitraries, for the equation connecting the two differentials now becomes

$$\frac{dz}{\sqrt{z(z-1)(z-z_0)}} = \frac{\text{constant} \times (x-x_1)\,dx}{\sqrt{x(x-1)(xX_1^2 - z_0 X_2^2)}},$$

where z_0 is one arbitrary, and the coefficients of X_1 and X_2 involve another.

The equation of transformation is here the same as for the cubic transformation of an elliptic integral; as, however, it is to be used for a different purpose, it is convenient to put it in a slightly altered form.

I shall therefore write

$$z : z-1 : 1$$

$$:: x(x+2t+t^2)^2 : (x-1)(x-t^2)^2 : \left[(1+2t)x+t^2\right]^2.$$

This gives

$$\frac{dz}{dx} = \frac{(x+2t+t^2)(x-t^2)\left[(1+2t)x-(2t+t^2)\right]}{\left[(1+2t)x+t^2\right]^3},$$

and hence

$$\frac{dz}{\sqrt{z(z-1)(z-z_0)}} = \frac{\left[(1+2t)x-(2t+t^2)\right]dx}{\sqrt{x(x-1)\left[x(x+2t+t^2)^2 - z_0\left\{(1+2t)x+t^2\right\}^2\right]}}.$$

The general form of hyperelliptic integral, therefore, which can be expressed, by cubic transformations, as the sum of two elliptic integrals, is

$$\int \frac{(a+bx)\,dx}{\sqrt{x(x-1)\left[x(x+2t+t^2)^2 - z_0\left\{(1+2t)x+t^2\right\}^2\right]}},$$

where t and z_0 are arbitrary.

It is worth observing that, from the properties of the equation for the cubic transformation of elliptic integrals, this may be expressed

in the form

$$\int \frac{(a+bx)\,dx}{\sqrt{x(x-1)(x-\mathrm{sn}^2 u)\left[x-\mathrm{sn}^2\left(u+\frac{2\omega}{3}\right)\right]\left[x-\mathrm{sn}^2\left(u+\frac{4\omega}{3}\right)\right]}},$$

2ω being any complete period of the elliptic function.

As regards the second transformation, it seemed natural to suppose that the two cubics $x(x-1)$ and $xX_1^2-z_0X_2^2$ into which the sextic under the square root naturally divides itself in the above analysis, would interchange parts, and I have found that such is the case.

This is perhaps most easily verified as follows. The equation for the second substitution

$$z=\frac{U}{V}$$

is, by supposition, to have roots 0, 1, ∞ for a particular value of s; hence, by a linear transformation of z, it can be thrown into the form

$$z=\frac{x^3+ax+b}{x(x-1)},$$

and it is to be shown that a and b can be so determined that this equation is consistent with

$$z-z_1=\frac{(x-x_1)(x-x_1')^2}{x(x-1)},$$

$$z-z_2=\frac{(x-x_2)(x-x_2')^2}{x(x-1)},$$

$$z-z_3=\frac{(x-x_3)(x-x_3')^2}{x(x-1)},$$

where $$(x-x_1)(x-x_2)(x-x_3)\equiv xX_1^2-z_0X_2^2.$$

Now $$(x^3+ax+b)(x_1^2-x_1)-(x^3-x)(x_1^3+ax_1+b)=0,$$

after casting out the factor $x-x_1$, has equal roots when

$$x_1^4+(2a+4b)x_1^3+(a^2-6b)x_1^2+(2ab+4b)x_1+b^2=0.$$

It is then to be shown that this equation is the same as

$$(x-x_0)\left[x(x+2t+t^2)^2-z_0\{(1+2t)x+t^2\}^2\right]=0.$$

A comparison of the coefficients, which is somewhat tedious but in

no way difficult, shows that the four equations of conditions, to which the identity of these equations leads, can be satisfied by the following values of a, b and x_0, viz.:—

$$a = t^2 + 2t(1-z_0) - z_0,$$

$$b = -z_0 t^2,$$

$$x_0 = z_0.$$

The second substitution is therefore

$$z = \frac{x^2 + x\left[t^2 + 2t(1-z_0) - z_0\right] - z_0 t^2}{x(x-1)}.$$

Differentiating and simplifying, this leads to

$$\frac{dz}{dx} = \frac{(x+t)\left[x^2 - (2+t)x^2 + z_0\left\{(2t+1)x - t\right\}\right]}{x^2(x-1)^2},$$

and, since again dz/dx only vanishes at the branch-points, this must be

$$\frac{dz}{dx} = \frac{(x+t)(x-x_1')(x-x_2')(x-x_3')}{x^2(x-1)^2}.$$

Hence, finally,

$$\frac{dz}{\sqrt{(z-z_1)(z-z_2)(z-z_3)}} = \frac{(x+t)\,dx}{\sqrt{x(x-1)(x-x_1)(x-x_2)(x-x_3)}}$$

$$= \frac{(x+t)\,dx}{\sqrt{x(x-1)\left[x(x+2t+t^2)^2 - z_0\left\{(1+2t)x + t^2\right\}^2\right]}}.$$

Collecting the results that have been obtained for this case, they may be stated thus:—

The integrals of the first species, belonging to the equation

$$s^2 = x(x-1)\left[x(x+2t+t^2)^2 - z_0\left\{(1+2t)x + t^2\right\}^2\right]$$

can always be transformed into the sum of two elliptic integrals by the substitutions

$$z = \frac{x(x+2t+t^2)^2}{\left[(1+2t)x + t^2\right]^2}$$

and

$$z = \frac{x^2 + t^2 x + 2t - z_0\left[x(1+2t) + t^2\right]}{x(x-1)},$$

and any form for which such a reduction can be effected by a cubic substitution can be transformed to the given one by a linear substitution.

Quintic Transformation.

For this case the equation for the quintic transformation of elliptic integrals is of no assistance, as it corresponds to an essentially different form of Riemann's surface.

The problem of finding the equation of transformation may be stated thus:—

To determine p, q, p', q' as functions of two parameters, so that

$$x(x^2+px+q)^2-(x-1)(x^2+p'x+q')^2$$

shall be a perfect square.

By a linear integral substitution for x, this may be thrown into the more symmetrical form

$$(x+\alpha)(x^2+Ax+1)^2-(x+\beta)(x^2+Bx+1)^2 = \text{perfect square.}$$

Taking $(\alpha-\beta)(ux^2+vx+1)^2$ for the perfect square, and equating coefficients, there results

$$2\frac{A-B}{\alpha-\beta} = u^2-1,$$

$$\frac{A\alpha-B\beta}{\alpha-\beta} = v,$$

$$\frac{A^2-B^2}{\alpha-\beta} = 2v(u-1),$$

$$\frac{A^2\alpha-B^2\beta}{\alpha-\beta} = v^2-(u-1)^2.$$

The solution of these equations presents no difficulty, and the only irrational quantity which appears is the square root of $v^2-(u+1)^2$. Writing w for this, the values at once obtained are

$$A+B = \frac{4v}{u+1}, \qquad \alpha+\beta = \frac{4v}{(u+1)^2},$$

$$A-B = 2w\frac{u-1}{u+1}, \qquad \alpha-\beta = \frac{4w}{(u+1)^2}.$$

The coefficients A, B, α, β may be expressed rationally in terms of two parameters, conveniently, by taking

$$u+1 = \frac{2}{st}, \qquad v = \frac{1}{s^2} + \frac{1}{t^2},$$

giving

$$w = \frac{1}{t^2} - \frac{1}{s^2},$$

and immediately leading to

$$\alpha = s^2, \qquad A = 2\frac{t}{s} + \frac{1}{t^2} - \frac{1}{s^2},$$

$$\beta = t^2, \qquad B = 2\frac{s}{t} + \frac{1}{s^2} - \frac{1}{t^2}.$$

The equation of transformation can then be written

$$z : z-1 : 1 :: (x+s^2)\left[x^2 + \left(2\frac{t}{s} + \frac{1}{t^2} - \frac{1}{s^2}\right)x + 1\right]^2$$
$$: (x+t^2)\left[x^2 + \left(2\frac{s}{t} + \frac{1}{s^2} - \frac{1}{t^2}\right)x + 1\right]^2$$
$$: (s^2-t^2)\left[\left(\frac{2}{st} - 1\right)x^2 + \left(\frac{1}{s^2} + \frac{1}{t^2}\right)x + 1\right]^2.$$

On differentiating and simplifying, this gives

$$\frac{-1}{(z-1)^3}\frac{dz}{dx} = (s^2-t^2)\,\frac{x^2 + \left(2\frac{t}{s} + \frac{1}{t^2} - \frac{1}{s^2}\right)x + 1}{(x+t^2)^2\left[x^2 + \left(2\frac{s}{t} + \frac{1}{s^2} - \frac{1}{t^2}\right)x + 1\right]^3}$$
$$\times\left[\left(\frac{2}{ts} - 1\right)x^2 + \left(\frac{1}{s^2} + \frac{1}{t^2}\right)x + 1\right]$$
$$\times\left[\left(\frac{2}{ts} - 1\right)x^2 + \left(\frac{1}{s^2} + \frac{1}{t^2}\right)(2ts-1)\,x + 4ts - 3\right].$$

The values of x at the remaining two branch-points are therefore given by

$$\left(\frac{2}{ts} - 1\right)x^2 + \left(\frac{1}{t^2} + \frac{1}{s^2}\right)(2ts-1)\,x + 4ts - 3 = 0;$$

and, if r is a root of this equation, the equation of transformation

may be written

$$z : z-1 : 1 : z-z_r$$

$$:: (x+s^2) \times \text{square} : (x+t^2) \times \text{square} : (s^2-t^2) \times \text{square}$$

$$: (x-r)^2 (x^3+ax^2+bx+c),$$

where a, b, c are rationally expressible in terms of r, s, t.

It may be noticed that the coefficients which are rational in s and t are also rational in ts and t/s, and that, writing

$$ts = r, \quad \frac{t}{s} = \sigma,$$

the equation defining r takes the simpler form

$$(2-r)\, r^3 + \left(\sigma + \frac{1}{\sigma}\right)(2r-1)\, r + r\,(4r-3) = 0,$$

which is quadratic in each variable; but I have not been able to determine whether the three quantities r, σ, r can be expressed as rational functions of two suitable chosen ones.

The hyperelliptic integral which this transformation immediately leads to is

$$\int \frac{(mx+n)\, dx}{\sqrt{(x+s^2)(x+t^2)(x^3+ax^2+bx+c)}},$$

where a, b, c are the determinate algebraical functions of s and t just introduced, and to this form every hyperelliptic integral can be brought which is reducible to two elliptic integrals by quintic transformations.

Transformations of Even Order.

When the transformation is of even order the product

$$(U-\alpha V)(U-\beta V)(U-\gamma V)(U-\delta V)$$

may be the product of a sextic and a perfect square in two ways. Either (i.) the first two factors may be perfect squares, while the third and fourth consist of the products of a quadratic and quartic, each with a square; or (ii.) the first factor may be a perfect square, and each of the others the product of a quadratic and a square.

The first case is clearly equivalent to the combination of a quadratic transformation of the elliptic integral into another elliptic integral, and then a transformation of this by a substitution of

degree $\frac{n}{2}$ into a hyperelliptic integral; and hence is not to be considered an independent transformation of order n. The second case need therefore alone be treated.

Considerations exactly parallel with those for a transformation of uneven order give, for the form of the equation of transformation,

$$z : z-1 : 1 : z-z_0 : z-z_1$$

$$:: X_1^2 : (x-1)(x-\kappa)\,X_2^2 : xX_3^2 : (x-\lambda)(x-\mu)\,X_4^2 : X_5\,(x-x_1)^2,$$

where X_1 is an integral function of x of degree $n/2$; X_2, X_3, X_4 are integral functions of degree $\frac{1}{2}n-1$, and X_5 is of degree $n-2$; the coefficients of the leading terms of X_1, X_2, X_4, X_5 are all unity (or the same), and the constants are all algebraical functions of two arbitraries. Supposing the coefficients so determined, then

$$\frac{dz}{dx} = \text{constant} \times \frac{X_1 X_2 X_4 (x-x_1)}{x^2 X_3^3},$$

and

$$\frac{dz}{\sqrt{z\,(z-1)(z-z_0)}} = \frac{\text{constant} \times (x-x_1)\,dx}{\sqrt{x\,(x-1)(x-\kappa)(x-\lambda)(x-\mu)}}.$$

Quartic Transformation.

In this case X_1 is a quadratic, and X_2, X_3, X_4 are linear factors. As in the case of the quintic transformation, the coefficients are most easily determined as functions of two arbitraries by performing a linear transformation on x, so as to make the forms more symmetrical. In particular, I suppose the variable so taken that the roots of $X_1=0$ are zero and infinity, and for convenience write $1/(1-z)$ for z. The equation of transformation takes the form

$$z : z-1 : 1 :: (x^2+2ax+a')(x+\alpha)^2 : (x^2+2bx+b')(x+\beta)^2 : Cx^2.$$

This gives

$$a'\alpha^2 - b'\beta^2 = 0,$$

$$a-b = \beta-\alpha,$$

$$a\alpha^2 - b\beta^2 = b'\beta - a'\alpha,$$

$$4\,(a\alpha - b\beta) + a' - b' + \alpha^2 - \beta^2 = C,$$

or

$$a' = v\beta^2, \quad b' = v\alpha^2, \quad a = \beta\frac{v\alpha+\beta}{\alpha+\beta}, \quad b = \alpha\frac{\alpha+v\beta}{\alpha+\beta},$$

$$C = \frac{(1-v)(\alpha-\beta)^2}{\alpha+\beta},$$

where v is a new arbitrary.

Writing now ax for x and u for β/α, the equation becomes

$$s : s-1 : 1$$

$$:: (x+1)^2\left(x^2+2u\frac{u+v}{u+1}x+vu^2\right) : (x+u)^2\left(x^2+2\frac{uv+1}{u+1}x+v\right)$$

$$: \frac{(1-v)(1-u)^2}{u+1}x^2.$$

On differentiating and reducing, this leads to

$$\frac{dz}{dx} = 2\frac{u+1}{(v-1)(u-1)^2}\frac{(x+1)(x+u)\left(x^2+\frac{(v-1)u}{u+1}x-vu\right)}{x^3}.$$

The values of x at the other two branch-points are therefore given by

$$x^2+\frac{(v-1)u}{u+1}x-uv=0.$$

If s, t are the roots of this equation,

$$u=\frac{s+t-st}{1-s-t},$$

$$v=-\frac{st(1-s-t)}{s+t-st},$$

and all the coefficients of the equation of transformation are rational in s and t. When they are so expressed, the equation in its complete form is

$$z : \left(x^2+2\frac{s^2+st+t^2}{1-s-t}x-\frac{st(s+t-st)}{1-s-t}\right)(x+1)^2$$

$$:: z-1 : \left(x^2+2(1-s-t)x-\frac{st(1-s-t)}{s+t-st}\right)\left(x+\frac{s+t-st}{1-s-t}\right)^2$$

$$:: 1 : \frac{(s+t)(1-2s-2t+st)^3}{(s+t-st)(1-s-t)^2}x^2$$

$$:: z-z_s : \left(x^2+2\frac{1-t+t^2}{1-s-t}x-\frac{t(s+t-st)}{s(1-s-t)}\right)(x-s)^2$$

$$:: z-z_t : \left(x^2+2\frac{1-s+s^2}{1-s-t}x-\frac{s(s+t-st)}{t(1-s-t)}\right)(x-t)^2.$$

The sextic function $f(x)$, such that $\int (a+bx)\,dx/\sqrt{f(x)}$ is reducible

to elliptic integrals by a quartic transformation, may then be written in the form

$$\left[x^2+2\,(s^2+st+t^2)\,x-stC\right]\left[x^2+2\,(1-s+s^2)\,x-\frac{s}{t}\,C\right]$$
$$\times\left[x^2+2\,(1-t+t^2)\,x-\frac{t}{s}\,C\right],$$

where $$C\equiv(s+t-st)(1-s-t).$$

For transformations of a higher order than the fifth, the relation between z and x becomes very much more complicated, as it appears to be then impossible to express it in a form in which the coefficients are rational functions of the parameters. The problem, too, of determining in each case the invariant character of the sextic, corresponding to the vanishing of the skew invariant for the case of the quadratic substitution, is undoubtedly one of considerable difficulty; this I have made no attempt to solve.

On the Analytical Theory of the Congruency. *By* Professor Cayley.

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If the lines of a congruency are considered as issuing from the several points of a surface, or say as the quasi-normals of a surface, then the fundamental geometrical theory is established by an analysis closely similar to that for the theory of the curvature of a surface; viz., it is shown that each quasi-normal is intersected by two consecutive quasi-normals in two points respectively (corresponding to the centres of curvature), or say in two foci; we have on the surface two series of curves of quasi-curvature—only these do not in general intersect at right angles; the intersecting quasi-normals form two series of developable surfaces, each touching the surface of centres (or focal surface), along its cuspidal edge, &c.; and, in particular, each quasi-normal is a bitangent of the focal surface, touching it at the two foci respectively.

But the analysis assumes a very different form if we consider the congruency by itself, without thus connecting it with a surface

Regarding the congruency as determined by means of two equations,

$$U(a, b, c, f, g, h) = 0, \quad V(a, b, c, f, g, h) = 0,$$

between the six coordinates of a line ($af+bg+ch=0$, as usual), I take (a, b, c, f, g, h) for the coordinates of a particular line of the congruency, $(a_1, b_1, c_1, f_1, g_1, h_1)$ for those of a consecutive line. Denoting, for shortness, the derived functions $(\partial_a, \partial_b, \partial_c, \partial_f, \partial_g, \partial_h)\, U$ by F, G, H, A, B, C, and similarly those of V by F', G', H', A', B', C', we have

$$a_1F + b_1G + c_1H + f_1A + g_1B + h_1C = 0,$$

$$a_1F' + b_1G' + c_1H' + f_1A' + g_1B' + h_1C' = 0;$$

viz., the consecutive line $(a_1, b_1, c_1, f_1, g_1, h_1)$ belongs to the linear congruency defined by these two equations.

Forming with these a linear combination

$$(\lambda F+\mu F')\, a_1 + (\lambda G+\mu G')\, b_1 + (\lambda H+\mu H')\, c_1 + (\lambda A+\mu A')\, f_1 + (\lambda B+\mu B')\, g_1 + (\lambda C+\mu C')\, h_1 = 0,$$

we may determine the ratio $\lambda:\mu$ by the equation

$$(\lambda A+\mu A')(\lambda F+\mu F') + (\lambda B+\mu B')(\lambda G+\mu G') + (\lambda C+\mu C')(\lambda H+\mu H') = 0,$$

that is

$$\lambda^2 (AF+BG+CH) + \lambda\mu\,(AF'+BG'+CH'+FA'+GB'+HC') + \mu^2 (A'B'+B'G'+C'H') = 0;$$

we have thus two values of $\lambda:\mu$; and, denoting the corresponding values of $\lambda F+\mu F'$, ... $\lambda C+\mu C'$ by $(f_2, g_2, h_2, a_2, b_2, c_2)$ and $(f_3, g_3, h_3, a_3, b_3, c_3)$ respectively, we have

$$a_2f_2 + b_2g_2 + c_2h_2 = 0, \quad a_3f_3 + b_3g_3 + c_3h_3 = 0,$$

and

$$a_1f_2 + b_1g_2 + c_1h_2 + f_1a_2 + g_1b_2 + h_1c_2 = 0,$$

$$a_1f_3 + b_1g_3 + c_1h_3 + f_1a_3 + g_1b_3 + h_1c_3 = 0;$$

viz., we have thus two lines $(a_2, b_2, c_2, f_2, g_2, h_2)$, $(a_3, b_3, c_3, f_3, g_3, h_3)$, not in general meeting each other, each of which is met by the line $(a_1, b_1, c_1, f_1, g_1, h_1)$; say, for shortness, the lines (a, b, c, f, g, h), $(a_1, b_1, c_1, f_1, g_1, h_1)$, $(a_2, b_2, c_2, f_2, g_2, h_2)$, $(a_3, b_3, c_3, f_3, g_3, h_3)$ are the lines 0, 1, 2, 3 respectively.

We may, in the foregoing investigation, substitute, for the coordi-

nates of the line 1, those of the line 0; and it hence appears—what is indeed obvious—that the line 0 meets each of the lines 2 and 3. Supposing now that the lines 0 and 1 meet each other, that is, that we have

$$af_1+bg_1+ch_1+fa_1+gb_1+hc_1 = 0,$$

then it is clear that the line 1 must pass through the intersection of the lines 0, 2, or else through the intersection of the lines 0, 3; in fact, if 0 and 1 intersect in a point not on the line 2 or 3, then we have the line 0 as a line passing through this point and meeting each of the lines 2 and 3; and also the line 1 as a line passing through this point and meeting each of the lines 2 and 3; that is, the lines 0 and 1 would be one and the same line.

It thus appears that, considering the line 0 as given, we have two lines 2 and 3 each meeting this line, say in the points P_2 and P_3 respectively; and that, this being so, the consecutive line 1 meets the line 0 either in the point P_2 or else in the point P_3, viz., that there are two consecutive lines 1, say 1_2 and 1_3, meeting the line 0 in the points 2 and 3 respectively. These points are thus given as the intersections of the line (a, b, c, f, g, h) with the lines $(a_2, b_2, c_2, f_2, g_2, h_2)$, $(a_3, b_3, c_3, f_3, g_3, h_3)$ respectively; viz., supposing that $\lambda : \mu$ is determined by the above-mentioned quadric equation, and calling its roots $\lambda_2 : \mu_2$ and $\lambda_3 : \mu_3$, then we have

$$a_2, b_2, c_2, f_2, g_2, h_2 = \lambda_2 A+\mu_2 A', \ldots, \lambda_2 H+\mu_2 H',$$

$$a_3, b_3, c_3, f_3, g_3, h_3 = \lambda_3 A+\mu_3 A', \ldots, \lambda_3 H+\mu_3 H'.$$

For the point 2, we have

$$\begin{array}{llll} . & hy-gz+aw=0, & . & h_2y-g_2z+a_2w=0, \\ -hx & . \quad +fz+bw=0, & -h_2x & . \quad +f_2z+b_2w=0, \\ gx-fy & . \quad +cw=0, & g_2x-f_2y & . \quad +c_2w=0, \\ -ax-by-cz & . \quad =0, & -a_2x-b_2y-c_2z & . \quad =0, \end{array}$$

each set of four equations being equivalent to two equations, in virtue of the relations $af+bg+ch=0$, $a_2f_2+b_2g_2+c_2h_2=0$ respectively. There is no completely symmetrical expression for the values of x, y, z, w; according as we derive them from the first equations, the second equations, the third equations, or the fourth equations of each set, we

obtain

$$\begin{aligned} x:y:z:w = \quad \Theta_1 \quad &: ag_2-ga_2 : ah_2-ha_2 : gh_2-hg_2, \\ = bf_2-fb_2 : \quad \Theta_2 \quad &: bh_2-hb_2 : hf_2-fh_2, \\ = cf_2-fc_2 : cg_2-gc_2 : \quad \Theta_3 \quad &: fg_2-gf_2, \\ = bc_2-cb_2 : ca_2-ac_2 &: ab_2-ba_2 : \quad \Theta; \end{aligned}$$

where

$$\begin{aligned} \Theta_1 &= -fa_2-bg_2-ch_2, \quad = af_2+gb_2+hc_2, \\ \Theta_2 &= -af_2-gb_2-ch_2, \quad = fa_2+bg_2+hc_2, \\ \Theta_3 &= -af_2-bg_2-hc_2, \quad = fa_2+gb_2+ch_2, \\ \Theta &= -af_2-bg_2-ch_2, \quad = fa_2+gb_2+hc_2. \end{aligned}$$

For the point 3, we have, of course, the same formulæ, with the suffix 3 instead of 2.

Notes on Dualistic Differential Transformations. *By* E. B. Elliott. Received and read March 10th, 1892.

In these notes I have found it impossible not to introduce much that is well known. I believe, however, that no one else has called attention to the coincidences with established theories, which give completeness to the later results of the first section, and clues towards completing those of the second; and I am not aware that anyone has spent time upon the case of three independent variables, considered in Section III.

I. *Transformation of Ordinary Differential Expressions.*

If, as usual, p denote $\frac{dy}{dx}$, and p' denote $\frac{dy'}{dx'}$, the transformation for x and y, in terms of x' and y',

$$\frac{ax+hy+g}{-p'} = \frac{hx+by+f}{1} = \frac{gx+fy+c}{p'x'-y'},$$

i.e.,
$$\frac{x}{-Ap'+H+G(p'x'-y')} = \frac{y}{-Hp'+B+F(p'x'-y')} = \frac{1}{-Gp'+F+C(p'x'-y')},$$

where $\begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix}$ and $\begin{vmatrix} A, & H, & G \\ H, & B, & F \\ G, & F, & C \end{vmatrix}$ are non-vanishing reciprocal determinants, is one possessing the quality of duality; in other words, x', y', x, y, p may replace x, y, x', y', p' in the formulæ given. The transformation, in fact, effects polar reciprocation with regard to the conic

$$ax^2+2hxy+by^2+2gx+2fy+c = 0.$$

In general the transformed expressions, for second and higher derivatives of y with regard to x, involve the variables x', y' themselves as well as derivatives. This is not the case, however, if $h = 0$ and $b = 0$, *i.e.*, if the base conic be a parabola with axis parallel to $x = 0$. In such a case

$$\frac{d^2y}{dx^2}\frac{d^2y'}{dx'^2} = \frac{a^3}{f^2}.$$

Since we can at any time re-introduce the right power of $-\frac{a}{f}$ as a factor by mere consideration of dimensions, we lose no real generality by putting $a = 1$, $f = -1$, and so taking as our base parabola

$$x^2-2y+2gx+c = 0.$$

The formulæ of transformation then become

$$x = p'-g, \quad y = gp'+B+p'x'-y',$$

$$x' = p-g, \quad y' = gp+B+px-y,$$

where $B = c-g^2$. It is, of course, generally convenient to take g and B zero.

We notice at once that $y_2, y_3, y_4, \ldots$, the second and higher derivatives of y with regard to x, are merely $\frac{dx'}{dx}, \frac{d^2x'}{dx^2}, \frac{d^3x'}{dx^3}, \ldots$, while

$y_2', y_3', y_4', \ldots$, those of y' with regard to x', are $\frac{dx}{dx'}, \frac{d^2x}{dx'^2}, \frac{d^3x}{dx'^3}, \ldots$.

Thus the transformation of second and higher derivatives amounts simply to the interchange of dependent and independent variables in a set of first and higher derivatives. Now, of this interchange a complete theory is at our disposal. It tells us that

$$y_2 = \frac{1}{y_2'},$$

and that

$$\frac{d}{dx} = \frac{1}{y_2'}\frac{d}{dx'};$$

so that

$$y_n = \left(\frac{1}{y_2'}\frac{d}{dx'}\right)^{n-2}\left(\frac{1}{y_2'}\right).$$

To transform any function of the derivatives, it is not, however, necessary to substitute for all derivatives y_n separately the values thus given. We have only to adapt as above a theorem of Mr. Leudesdorf's (*Proc. Lond. Math. Soc.*, Vol. XVII., p. 333). The theorem at which we arrive from it is that, if

$$U \equiv 3y_3^2\frac{d}{dy_4} + 10y_3y_4\frac{d}{dy_5} + (15y_3y_5 + 10y_4^2)\frac{d}{dy_6} + \ldots$$

$$\equiv \sum_{m\geqslant 3}\sum_{r\geqslant 3}^{r\geqslant m}(m)_{r-1}\, y_r y_{m-r+3}\frac{d}{dy_{m+1}},$$

where $(m)_{r-1}$ denotes the number of combinations of m things $r-1$ together, and, if f denote a homogeneous isobaric function of second and higher derivatives, of degree i and weight w,

$$f(y_2', y_3', y_4', \ldots) = (-1)^i y_2^{-w} e^{-1/y_2 \cdot U} f(-y_2, y_3, y_4, \ldots).$$

Thus, if any differential expression is the sum of such terms as

$$\phi(x', y', p')\, f(y_2', y_3', y_4', \ldots),$$

where f is homogeneous and isobaric, its dual consists of such corresponding terms as

$$(-1)^i \phi(p-g, gp+B+px-y, x+g)\, y_2^{-w} e^{-1/y_2 \cdot U} f(-y_2, y_3, y_4, \ldots).$$

Another remark, of very comprehensive importance, to which the simple fact we have noticed leads, is that the whole theory of recipro-

cants has its direct application to our dualistic transformation. If, in fact,

$$R(y_1, y_2, y_3, \ldots)$$

be any reciprocant whatever, pure or mixed, and if in it we replace each derivative by the next higher, we thus obtain

$$R(y_2, y_3, y_4, \ldots),$$

a function of the second and higher derivatives which, but for a power of y_2 as factor, is its own dual. In other words, such functions are the criteria of curves whose reciprocals, with regard to a parabola with axis parallel to that of y_1, constitute the same family.

In particular, functions of $y_3, y_4, y_5, \ldots$ without y_2, or with only odd or only even powers of y_2, which U annihilates, are such functions. They correspond to pure reciprocants, or sums of products of pure reciprocants and odd or even powers of the first derivative.

II. *Partial Differential Expressions with Two Independent Variables.*

1. In this, the case of ordinary three-dimensional geometry, let, as usual, p, q denote $\frac{dz}{dx}$, $\frac{dz}{dy}$.

Two classes of transformations, expressing x, y, z homographically in terms of p', q', $p'x'+q'y'-z'$, have the dualistic property, and correspond to two kinds of geometrical reciprocity of surfaces. The homographic formulæ have been given by Chasles (*Aperçu Historique*, Note xxx.) for the first class, and he has also indicated the second class by means of its most important case. The classes are (A) that of transformations in which the coordinates of a point of a surface are expressed by means of a corresponding point on the surface which is the polar reciprocal of the former surface with regard to a fixed quadric, and (B) the class in which the coordinates of a point of a surface are expressed by means of the coordinates of the point of contact with the surface, which it envelopes, of a plane passing through the first point and orthogonal to the direction of its motion on an instantaneous screw of given pitch and axis.

(A) The formulæ given by Chasles for the first class are in effect

$$\frac{ax+hy+gz+l}{-p'} = \frac{hx+by+fz+m}{-q'} = \frac{gx+fy+cz+n}{1} = \frac{lx+my+nz+d}{p'x'+q'y'-z'},$$

i.e.,

$$\frac{x}{-Ap'-Hq'+G+L(p'x'+q'y'-z')}$$

$$=\frac{y}{-Hp'-Bq'+F+M(p'x'+q'y'-z')}$$

$$=\frac{z}{-Gp'-Fq'+C+N(p'x'+q'y'-z')}$$

$$=\frac{1}{-Lp'-Mq'+N+D(p'x'+q'y'-z')},$$

where $\begin{vmatrix} a, & h, & g, & l \\ h, & b, & f, & m \\ g, & f, & c, & n \\ l, & m, & n, & d \end{vmatrix}$ and $\begin{vmatrix} A, & H, & G, & L \\ H, & B, & F, & M \\ G, & F, & C, & N \\ L, & M, & N, & D \end{vmatrix}$ are non-vanishing reciprocal determinants.

Of these the Monge–De Morgan formulæ

$$x=p',\quad y=q',\quad z=p'x'+q'y'-z'$$

are a particular case.

(B) The general formulæ for the second class of dualistic transformations are found by identifying

$$p'(\xi-x')+q'(\eta-y')-(\zeta-z')=0$$

with $\{vl+\omega(m\bar{z}-n\bar{y})\}(\xi-x)+\{vm+\omega(n\bar{x}-l\bar{z})\}(\eta-y)$
$$+\{vn+\omega(l\bar{y}-m\bar{x})\}(\zeta-z)=0,$$

where $\bar{x}$, $\bar{y}$, $\bar{z}$ stand for $x-a$, $y-b$, $z-c$, where

$$\frac{x-a}{l}=\frac{y-b}{m}=\frac{z-c}{n}$$

is the axis of the screw, and $v\omega^{-1}$ its pitch. They are of the form

$$\frac{-p'}{mz-ny+\lambda}=\frac{-q'}{nx-lz+\mu}=\frac{1}{ly-mx+\nu}=\frac{z'-p'x'-q'y'}{\lambda x+\mu y+\nu z},$$

which may be written

$$\frac{x}{\nu q'+\mu-l(z'-p'x'-q'y')}=\frac{y}{-\nu p'-\lambda-m(z'-p'x'-q'y')}$$
$$=\frac{z}{\mu p'-\lambda q'-n(z'-p'x'-q'y')}=\frac{1}{lp'+mq'-n}.$$

The only restrictions upon the constants are that they be finite and such that $l\lambda+m\mu+n\nu$, *i.e.*, $v\omega^{-1}$, do not vanish.

The included transformation discussed by Chasles is

$$x = q', \quad y = -p', \quad z = z'-p'x'-q'y'.$$

2. Regarding Class (A) first, I have considered the transformation of second and higher derivatives only for the included cases where x', y', z' are linear in $p, q, px+qy-z$. With altered notation we have

$$x = ap'+hq'-g,$$
$$y = hp'+bq'-f,$$
$$z = gp'+fq'+c+p'x'+q'y'-z',$$

leading to

$$x' = ap+hq-g,$$
$$y' = hp+bq-f,$$
$$z' = gp+fq+c+px+qy-z.$$

We have at once

$$\frac{d(x, y)}{d(x', y')} = (ab-h^2)(r't'-s'^2) = \frac{1}{(ab-h^2)(rt-s^2)},$$

where r, s, t, r', s', t' denote $\frac{d^2z}{dx^2}, \frac{d^2z}{dxdy}, \frac{d^2z}{dy^2}, \frac{d^2z'}{dx'^2}, \frac{d^2z'}{dx'dy'}, \frac{d^2z'}{dy'^2}$, so that

$$(ab-h^2)^2(rt-s^2)(r't'-s'^2) = 1.$$

We have also, since

$$\frac{du}{dx'} = \frac{d(u, y')}{d(x, y)} \cdot \frac{d(x, y)}{d(x', y')},$$

and

$$\frac{du}{dy'} = \frac{d(x', u)}{d(x, y)} \cdot \frac{d(x, y)}{d(x', y')},$$

the operator equivalents, sufficing to produce derivatives of any order from those of the order preceding,

$$\frac{d}{dx'} = \frac{1}{(ab-h^2)(rt-s^2)}\left\{(hs+bt)\frac{d}{dx}-(hr+bs)\frac{d}{dy}\right\},$$

$$\frac{d}{dy'} = \frac{1}{(ab-h^2)(rt-s^2)}\left\{(ar+hs)\frac{d}{dy}-(as+ht)\frac{d}{dx}\right\}.$$

By use of these or by the method of Forsyth's "Differential Equations," § 242, we readily obtain

$$r' = \frac{1}{(ab-h^2)^2}\cdot\frac{b^2r+2hbs+h^2t}{rt-s^2},$$

$$s' = -\frac{1}{(ab-h^2)^2}\cdot\frac{abr+(ab+h^2)s+hbt}{rt-s^2},$$

$$t' = \frac{1}{(ab-h^2)^2}\cdot\frac{a^2r+2ahs+h^2t}{rt-s^2}.$$

whence the derivation of the transformed values of higher derivatives is direct, though soon laborious.

3. The Monge–De Morgan case

$$x' = p, \quad y' = q, \quad z' = px+qy-z,$$

or the very slightly more general

$$x' = p-g, \quad y' = q-f, \quad z' = gp+fq+c+px+qy-z,$$

$$x = p'-g, \quad y = q'-f, \quad z = gp'+fq'+c+p'x'+q'y'-z',$$

which is that of reciprocation with regard to a paraboloid of revolution with its axis parallel to that of z, is of special interest; and here a contact with an existing theory affords, as in the corresponding case in Section I., a means of elaborating many interesting results.

We notice, in fact, that the second unaccented derivatives are merely $\frac{dx'}{dx}$, $\frac{dx'}{dy} \equiv \frac{dy'}{dx}$, $\frac{dy'}{dy}$, and that the corresponding accented derivatives are $\frac{dx}{dx'}$, $\frac{dx}{dy'} \equiv \frac{dy}{dx'}$, $\frac{dy}{dy'}$; and generally that the theory of the expression of the second and higher derivatives of z', with regard to x' and y' in terms of those of z with regard to x and y, is included in the theory of the interchange of the dependent and the independent pairs in functions of the first and higher partial derivatives of one pair of variables with regard to another pair of which they are regarded as functions. A paper by the present author on this latter theory will be found in the *Proc. Lond. Math. Soc.*, Vol. XXII., pp. 79, &c. It must be remembered that the theory whose application to the purposes of the present article is immediate is included in, but less general than, that developed in the paper referred

to. The fact that, as above noticed, $\frac{dx'}{dy} = \frac{dy'}{dx}$, has nothing corresponding to it in the paper quoted. The $\frac{d^{m+n-1}x'}{dx^{m-1}dy^n}$ and $\frac{d^{m+n-1}y'}{dx^m dy^{n-1}}$ of that paper have here both the same meaning, namely, $\frac{d^{m+n}z}{dx^m dy^n}$. Similarly $\frac{d^{m+n-1}x}{dx'^{m-1}dy'^n}$ and $\frac{d^{m+n-1}y}{dx'^m dy'^{n-1}}$ both mean $\frac{d^{m+n}z'}{dx'^m dy'^n}$.

For instance, what is given by the last result of the paper is that

$$4(z_{30}z_{12}-z_{21}^2)(z_{21}z_{03}-z_{12}^2)-(z_{30}z_{03}-z_{21}z_{12})^2,$$

the discriminant of $\left(h\frac{d}{dx}+k\frac{d}{dy}\right)^3 z$, is, but for a power of $rt-s^2$ as factor, unaltered by our dualistic transformation, *i.e.*, is the criterion of a family of surfaces whose polar reciprocals, with regard to a paraboloid with axis of revolution parallel to the axis of z, are surfaces of the same family. The fact is otherwise at once clear from the equalities, written down by the means already described,

$$z'_{30} = \{s^3z_{03}-3s^2tz_{12}+3st^2z_{21}-t^3z_{30}\}\div,$$

$$z'_{21} = \{-rs^2z_{03}+(2rst+s^3)z_{12}-(rt^2+2s^2t)z_{21}+st^2z_{30}\}\div,$$

$$z'_{12} = \{r^2sz_{03}-(2rs^2+r^2t)z_{12}+(2rst+s^3)z_{21}-s^2tz_{30}\}\div,$$

$$z'_{03} = \{-r^3z_{03}+3r^2sz_{12}-3rs^2z_{21}+s^3z_{30}\}\div,$$

the common denominator being $(rt-s^2)^3$.

The transformed expressions for higher derivatives are deduced by aid of the operator equivalences

$$\frac{d}{dx'} = \frac{1}{rt-s^2}\left\{t\frac{d}{dx}-s\frac{d}{dy}\right\},$$

$$\frac{d}{dy'} = \frac{1}{rt-s^2}\left\{r\frac{d}{dy}-s\frac{d}{dx}\right\}.$$

Elementary absolute self-dual functions are

$$(rt-s^2)^{-\frac{1}{2}}s, \quad (rt-s^2)^{-\frac{1}{2}}(r+t), \quad (rt-s^2)^{-\frac{1}{2}}(r-t),$$

of characters $-$, $+$, $-$ respectively; and, u, v being any two abso-

lute self-dual functions, another is

$$(rt-s^2)^{-\frac{1}{2}}\frac{d(u,v)}{d(x,y)}.$$

We may, for instance, conveniently take for u and v the two $\frac{r\pm t}{s}$ in the first place, and successively afterwards others derived from those two.

4. For the linear case of Class (B),

$$x' = hq+g,$$
$$y' = -hp+f,$$
$$z' = gp+fq+z-px-qy,$$

which is only in appearance more general than that of Chasles, we have $h^4(rt-s^2)(r't'-s'^2) = 1,$

and $$\frac{r'}{r} = \frac{s'}{s} = \frac{t'}{t} = -\frac{1}{h^2}\cdot\frac{1}{rt-s^2}.$$

We have also $$\frac{d}{dx'} = \frac{1}{h(rt-s^2)}\left\{r\frac{d}{dy}-s\frac{d}{dx}\right\},$$

and $$\frac{d}{dy'} = \frac{1}{h(rt-s^2)}\left\{s\frac{d}{dy}-t\frac{d}{dx}\right\}.$$

Hence, quite readily,

$$z'_{30} = \left\{-s^3z_{30}+3s^2rz_{21}-3sr^2z_{12}+r^3z_{03}\right\}\div,$$
$$z'_{21} = \left\{-s^2tz_{30}+(s^3+2rst)\,z_{21}-(2rs^2+r^2t)\,z_{12}+r^2sz_{03}\right\}\div,$$
$$z'_{12} = \left\{-st^2z_{30}+(2s^2t+rt^2)\,z_{21}-(2rst+s^3)\,z_{12}+rs^2z_{03}\right\}\div,$$
$$z'_{03} = \left\{-t^3z_{30}+3s^2tz_{21}-3st^2z_{12}+s^3z_{03}\right\}\div,$$

the common denominator of all four being $h^3(rt-s^2)^3$.

If h be unity,

$$(rt-s^2)^{-\frac{1}{2}}r, \quad (rt-s^2)^{-\frac{1}{2}}s, \quad (rt-s^2)^{-\frac{1}{2}}t$$

are elementary self-dual functions, absolute, and all of negative

character. From any two such, u and v say, we can at once derive another of higher order, viz.,

$$(rt-s^2)^{-1}\frac{d(u, v)}{d(x, y)}.$$

For instance, we may, in the first place, take $\frac{r}{s}$ and $\frac{t}{s}$ for u and v.

III. *Partial Differential Expressions with Three Independent Variables.*

1. If u is a function of three variables x, y, z, and if p, q, r denote now $\frac{du}{dx}, \frac{du}{dy}, \frac{du}{dz}$, dualistic formulæ of transformation analogous to those of Class (A), Section II., are with ease developed. In what follows attention is confined to those in which new independent variables x', y', z' are expressed linearly in terms of p, q, r and $px+qy+rz-u$. They are

$$\begin{aligned} x' &= ap+hq+gr-l, \\ y' &= hp+bq+fz-m, \\ z' &= gp+fq+cz-n, \\ u' &= lp+mq+nr+d+px+qy+rz-u. \end{aligned}$$

Without real loss of generality, we may put x, x', y, y', z, z' for $x+l, x'+l, y+m, y'+m, z+n, z'+n$, and write

$$\begin{aligned} x' &= ap+hq+gz, \\ y' &= hp+bq+fz, \\ z' &= gp+fq+cz, \\ u' &= px+qy+rz-u+d, \end{aligned}$$

of which the first three, otherwise written, are

$$\begin{aligned} \Delta p &= Ax'+Hy'+Gz', \\ \Delta q &= Hx'+By'+Fz', \\ \Delta r &= Gx'+Fy'+Cz', \end{aligned}$$

where A, B, C, F, G, H are the minors corresponding to a, b, c, f, g, h

in the non-va nishing determinant

$$\Delta \equiv \begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix}.$$

By differentiation of u', we have

$$p'dx'+q'dy'+r'dz' = xdp+ydq+zdr$$

$$= \frac{1}{\Delta}\{(Ax+Hy+Gz)\,dx'+(Hx+By+Fz)\,dy'+(Gx+Fy+Cz)\,dz'\};$$

from which it follows that

$$\Delta p' = Ax+Hy+Gz,$$
$$\Delta q' = Hx+By+Fz,$$
$$\Delta r' = Gx+Fy+Cz,$$

which, with

$$u = px+qy+rz-u'+d$$
$$= \frac{1}{\Delta}\Sigma.(Ax'+Hy'+Gz')(ap'+hq'+gr')-u'+d$$
$$= p'x'+q'y'+r'z'-u'+d,$$

show that the duality is complete.

2. Let u_{11}, u_{12}, ..., u'_{11}, u'_{12}, ... denote the second derivatives $\frac{d^2u}{dx^2}$, $\frac{d^2u}{dx\,dy}$, ..., $\frac{d^2u'}{dx'^2}$, $\frac{d^2u'}{dx'\,dy'}$, ...; and let

$$J = \begin{vmatrix} u_{11}, & u_{12}, & u_{13} \\ u_{21}, & u_{22}, & u_{23} \\ u_{31}, & u_{32}, & u_{33} \end{vmatrix},$$

a determinant whose minors will be called $J_{11}, J_{12}, \ldots$. Moreover J', J'_{11}, J'_{12}, ... are the like functions of accented derivatives.

By differentiation of x', y', z', we obtain

$$dx' = (au_{11}+hu_{12}+gu_{13})\,dx+(au_{12}+hu_{22}+gu_{23})\,dy+(au_{13}+hu_{23}+gu_{33})\,dz,$$
$$dy' = (hu_{11}+bu_{12}+fu_{13})\,dx+(hu_{12}+bu_{22}+fu_{23})\,dy+(hu_{13}+bu_{23}+fu_{33})\,dz,$$
$$dz' = (gu_{11}+fu_{12}+cu_{13})\,dx+(gu_{12}+fu_{22}+cu_{23})\,dy+(gu_{13}+fu_{23}+cu_{33})\,dz.$$

We have also, by differentiation of $\Delta p'$, $\Delta q'$, $\Delta r'$,

$$\Delta\,(u'_{11}dx'+u'_{12}dy'+u'_{13}dz') = A\,dx+H\,dy+G\,dz,$$
$$\Delta\,(u'_{12}dx'+u'_{22}dy'+u'_{23}dz') = H\,dx+B\,dy+F\,dz,$$
$$\Delta\,(u'_{13}dx'+u'_{23}dy'+u'_{33}dz') = G\,dx+F\,dy+C\,dz,$$

which give other linear expressions for dx', dy', dz', in terms of dx dy, dz. The identification of these with the former gives at once

$$\Delta J'\,(au_{11}+hu_{12}+gu_{13}) = AJ'_{11}+HJ'_{12}+GJ'_{13},$$
$$\Delta J'\,(hu_{11}+bu_{12}+fu_{13}) = AJ'_{12}+HJ'_{22}+GJ'_{23},$$
$$\Delta J'\,(gu_{11}+fu_{12}+cu_{13}) = AJ'_{13}+HJ'_{23}+GJ'_{33},$$

$$\Delta J'\,(au_{12}+hu_{22}+gu_{23}) = HJ'_{11}+BJ'_{12}+FJ'_{13},$$
$$\Delta J'\,(hu_{12}+bu_{22}+fu_{23}) = HJ'_{12}+BJ'_{22}+FJ'_{23},$$
$$\Delta J'\,(gu_{12}+fu_{22}+cu_{23}) = HJ'_{13}+BJ'_{23}+FJ'_{33},$$

$$\Delta J'\,(au_{13}+hu_{23}+gu_{33}) = GJ'_{11}+FJ'_{12}+CJ'_{13},$$
$$\Delta J'\,(hu_{13}+bu_{23}+fu_{33}) = GJ'_{12}+FJ'_{22}+CJ'_{23},$$
$$\Delta J'\,(gu_{13}+fu_{23}+cu_{33}) = GJ'_{13}+FJ'_{23}+CJ'_{33}.$$

These express for us u_{11}, u_{12}, ... in terms of J' and its minors. They give us, in fact,

$$\Delta^2 J'u_{11} = A^2J'_{11}+2AHJ'_{12}+2AGJ'_{13}+H^2J'_{22}+2HGJ'_{23}+G^2J'_{33},$$
$$\Delta^2 J'u_{22} = H^2J'_{11}+2HBJ'_{12}+2HFJ'_{13}+B^2J'_{22}+2BFJ'_{23}+F^2J'_{33},$$
$$\Delta^2 J'u_{33} = G^2J'_{11}+2GFJ'_{12}+2GCJ'_{13}+F^2J'_{22}+2FCJ'_{23}+C^2J'_{33},$$

$$\Delta^2 J\,u_{12} = AHJ'_{11}+(AB+H^2)\,J'_{12}+(AF+GH)\,J'_{13}+BHJ'_{22}$$
$$+(HF+BG)\,J'_{23}+FGJ'_{33},$$

$$\Delta^2 J'u_{13} = AGJ'_{11}+(AF+GH)\,J'_{12}+(AC+G^2)\,J'_{13}+HFJ'_{22}$$
$$+(CH+FG)\,J'_{23}+CGJ'_{33},$$

$$\Delta^2 J'u_{23} = GHJ'_{11}+(BG+HF)\,J'_{12}+(CH+FG)\,J'_{13}+BFJ'_{22}$$
$$+(BC+F^2)\,J'_{23}+CFJ'_{33}.$$

We may briefly express the facts arrived at by saying that the

linear transformation

$$\alpha = a\alpha' + h\beta' + g\gamma',$$
$$\beta = h\alpha' + b\beta' + f\gamma',$$
$$\gamma = g\alpha' + f\beta' + c\gamma'$$

produces $\frac{1}{J'}\{J'_{11}\alpha'^2 + J'_{22}\beta'^2 + J'_{33}\gamma'^2 + 2J'_{23}\beta'\gamma' + 2J'_{13}\gamma'\alpha' + 2J'_{12}\alpha'\beta'\}$,

as the equivalent of

$$u_{11}\alpha^2 + u_{22}\beta^2 + u_{33}\gamma^2 + 2u_{23}\beta\gamma + 2u_{13}\gamma\alpha + 2u_{12}\alpha\beta.$$

One conclusion hence is that

$$\Delta^2 JJ' = 1;$$

and this might at once have been written down by noticing that

$$\frac{d(x', y', z')}{d(x, y, z)} = \Delta J,$$

and

$$\frac{d(x, y, z)}{d(x', y', z')} = \Delta J'.$$

3. A few words as to the particular dualistic transformation

$$x' = p,$$
$$y' = q,$$
$$z' = r,$$
$$u' = px + qy + rz - u,$$

which, as in Section II., 3, may be given an apparently slightly more general form, will conclude the present paper. The formulæ of transformation of second derivatives become now

$$\frac{u_{11}}{J'_{11}} = \frac{u_{12}}{J'_{12}} = \frac{u_{13}}{J'_{13}} = \frac{u_{22}}{J'_{22}} = \frac{u_{23}}{J'_{23}} = \frac{u_{33}}{J'_{33}} = \frac{1}{J'} = J$$
$$= \frac{J_{11}}{u'_{11}} = \frac{J_{12}}{u'_{12}} = \frac{J_{13}}{u'_{13}} = \frac{J_{22}}{u'_{22}} = \frac{J_{23}}{u'_{23}} = \frac{J_{33}}{u'_{33}}.$$

As an example of conclusions arrived at it may be noticed that

every solution of an equation linear in u_{11}, u_{12}, &c., corresponds to a solution of a corresponding equation linear in J'_{11}, J'_{12}, &c.

In particular, to every spherical harmonic, or solution of

$$\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}+\frac{d^2u}{dz^2}=0,$$

corresponds a single solution of

$$\Sigma\left\{\frac{d^2u}{dx^2}\,\frac{d^2u}{dy^2}-\left(\frac{d^2u}{dx\,dy}\right)^2\right\}=0.$$

For instance, $u=(x^2+y^2+z^2)^{-\frac{1}{2}}=\rho^{-1}$ is a spherical harmonic. The corresponding solution of the dual equation last written is obtained from

$$x'=p=-x\rho^{-3},$$

$$y'=q=-y\rho^{-3},$$

$$z'=r=-z\rho^{-3},$$

$$u'=px+qy+rz-u=-2\rho^{-1}$$

$$=-2\,(x'^2+y'^2+z'^2)^{\frac{1}{2}}.$$

Thus $u'=\rho'^{\frac{1}{2}}$ is a solution as required.

More generally, let u_n be a spherical harmonic of degree n. It is homogeneous in x, y, z. A corresponding solution u' of the dual equation is given by

$$x'=\frac{du_n}{dx},$$

$$y'=\frac{du_n}{dy},$$

$$z'=\frac{du_n}{dz},$$

$$u'=(n-1).u_n$$

$$=\text{a homogeneous function of degree }\frac{n}{n-1}\text{ in }x',\ y',\ z'.$$

Thus the degrees n, n' of the spherical harmonic and its dual are connected by the relation $\frac{1}{n}+\frac{1}{n'}=1$. Moreover, to the relation

$n_1+n_2=-1$, connecting the positive and negative degrees of two associated spherical harmonics, corresponds the relation

$$3n_1'n_2'-2(n_1'+n_2')+1=0,$$

connecting the degrees of two associated solutions of the dual equation. Such two associated solutions differ by a factor which is a power of $\left(\frac{du'}{dx'}\right)^2+\left(\frac{du'}{dy'}\right)^2+\left(\frac{du'}{dz'}\right)^2$, where u' is either of the two.

On certain Quartic Curves of the Fourth Class and the Porism of the Inscribed and Circumscribed Polygon. *By* R. A. Roberts.

Received March 8th, 1892. Read March 10th, 1892.

In a former communication to the *Proceedings* of the Society, I showed (see Vol. XVI., p. 53) that for certain plane curves of the fourth degree there existed an infinite number of polygons simultaneously inscribed in and circumscribed about the curve; and I gave the means of determining the parameter in the equation of the curve in the same way as Professor Cayley has done for the similar problem of two conics, namely, by equating to zero determinants formed by the coefficients in the expansion of a radical expression. It may be worth while, however, making the investigation directly, with the use of the elliptic integrals by means of which the results are arrived at.

In the place referred to, I showed that the curves are unicursal and possessed of two cusps and a node. Slightly altering the notation, I write the curve now

$$(xy+zx-yz)^2-m^2z^2xy=0,$$

where xz, yz are the cusps, and xy is the node. This equation, it is easy to verify, is satisfied by taking

$$x=1-\theta^2-m\theta,\quad y=\theta^2(1-\theta^2-m\theta),\quad z=\theta^2,$$

where θ is a parameter. Hence, if a line

$$\lambda x+\mu y+\nu z=0$$

meet the curve, we have

$$\lambda(1-\theta^2-m\theta)+\mu\theta^2(1-\theta^2-m\theta)+\nu\theta^2=0,$$

from which we find that the four parameters of the points of intersection are connected by the relations

$$\Sigma\theta=-m, \quad \Sigma\frac{1}{\theta}=m.$$

Hence, if the tangent to the curve at the point θ meet the curve again at the points θ_1, θ_2, we have

$$\theta_1+\theta_2+2\theta=-m, \quad \frac{1}{\theta_1}+\frac{1}{\theta_2}+\frac{2}{\theta}=m;$$

whence, eliminating θ, we obtain

$$(\theta_1+\theta_2)^2+m(\theta_1+\theta_2)(1-\theta_1\theta_2)-(m^2+4)\theta_1\theta_2=0 \ldots\ldots\ldots(1),$$

which is consequently the relation connecting the parameters of two points, such that the line joining them touches the curve. Now this is of the well known form which occurs in the case of two conics; in fact, if we consider the conic $x=1$, $y=\theta^2$, $z=2\theta$, *i.e.*, $z^2-4xy=0$, the tangents at the points θ_1, θ_2 intersect in

$$x=1, \quad y=\theta_1\theta_2, \quad z=\theta_1+\theta_2,$$

so that (1) expresses that two tangents to $z^2-4xy=0$ intersect on the conic

$$z^2+mz(x-y)-(m^2+4)xy=0 \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

The problem is thus reduced to finding the conditions that it should be possible to inscribe an infinite number of polygons in (2), so that they should be circumscribed about $z^2-4xy=0$.

It will now be very convenient to make use of a result given by Mr. L. J. Rogers (*Proceedings*, Vol. XVI., p. 306).

Let
$$S=z^2+mz(x-y)-(m^2+4)xy,$$
$$S'=z^2-4xy;$$

then let us form first the discriminant of $\xi S'+S$, when we find

$$\{4\xi+m^2+4\}\{4\xi^2+(m^2+8)\xi+4\}.$$

Now, in order to render all the factors rational, we put

$$m^2=\frac{4(t-1)^2}{t},$$

when the discriminant becomes proportional to

$$\left(\xi+t+\frac{1}{t}-1\right)(\xi+t)\left(\xi+\frac{1}{t}\right).$$

But Mr. L. J. Rogers has shown that, if the discriminant is proportional to

$$(\xi-\xi_1)\left(\xi-\xi_1\frac{\mathrm{cn}^2 v}{\mathrm{dn}^2 v}\right)(\xi-\xi_1\,\mathrm{cn}^2 v),$$

the condition for a polygon of n sides is

$$v=\frac{2K}{n}.$$

In this case we have $\xi_1=-t$,

and $$\mathrm{cn}^2 v=\frac{1}{t^2},\quad \frac{\mathrm{cn}^2 v}{\mathrm{dn}^2 v}=\frac{1+t^2-t}{t^2};$$

whence $$k^2=\frac{1-\mathrm{dn}^2 v}{1-\mathrm{cn}^2 v}=\frac{t^2}{1+t^2}.$$

We thus have $$\mathrm{sn}\, v=\frac{\sqrt{(t^2-1)}}{t},\quad \mathrm{cn}\, v=\frac{1}{t},$$

$$\mathrm{dn}\, v=\frac{1}{\sqrt{(1+t^2-t)}} \qquad \text{.....(3).}$$

To find $\mathrm{sn}\, 2v$, we have

$$\mathrm{sn}\, 2v=\frac{2\,\mathrm{sn}\, v\,\mathrm{cn}\, v\,\mathrm{dn}\, v}{1-k^2\,\mathrm{sn}^4 v}=\frac{2\sqrt{\{(t-1)(t^2+1)\}}}{t\,(2t-1)},$$

and $$\mathrm{cn}\, 2v=\frac{1-2\,\mathrm{sn}^2 v+k^2\,\mathrm{sn}^4 v}{1-k^2\,\mathrm{sn}^4 v}=\frac{2-t}{t\,(2t-1)},$$

$$\mathrm{dn}\, 2v=\frac{1-2k^2\,\mathrm{sn}^2 v+k^2\,\mathrm{sn}^4 v}{1-k^2\,\mathrm{sn}^4 v}=\frac{1}{2t-1} \qquad \text{.....(4).}$$

Again, since $$\mathrm{sn}\, v\,\mathrm{sn}\, 3v=\frac{\mathrm{sn}^2 2v-\mathrm{sn}^2 v}{1-k^2\,\mathrm{sn}^2 v\,\mathrm{sn}^2 2v},$$

we get $$\mathrm{sn}\, 3v=\frac{3\sqrt{(t^2-1)}}{5t-4} \qquad \text{.....(5);}$$

whence $$\mathrm{cn}\, 3v=\frac{5-4t}{5t-4},\quad \mathrm{dn}\, 3v=\frac{4t^2-7t+4}{(5t-4)\sqrt{(t^2-t+1)}},$$

as can also be proved otherwise.

Again, we have

$$\operatorname{cn} 4v = \frac{1-2\operatorname{sn}^2 2v+k^2\operatorname{sn}^4 2v}{1-k^2\operatorname{sn}^4 2v}$$

$$= \frac{1-\dfrac{8(t-1)(t^3+1)}{t^2(2t-1)^2}+\dfrac{16t^3}{1+t^3}\dfrac{(t-1)^2(t^3+1)^2}{t^4(2t-1)^4}}{1-\dfrac{16t^3}{1+t^3}\dfrac{(t-1)^2(t^3+1)^2}{t^4(2t-1)^4}}$$

$$= \frac{-16t^3+33t^2-24t+8}{t(8t^3-24t^2+33t-16)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6).$$

We may now apply these expressions to finding the values of t, when n takes different values.

Let $n=3$; then $3v=2K$, and $\operatorname{sn} 3v=0$, which gives, from (5), $t=\pm 1$. Now these values are manifestly irrelevant, as, from (3), they actually correspond to $v=0$, and $v=2K$.

Thus we see that there are no curves simultaneously inscribed in and circumscribed about an infinite system of triangles.

Next, let $n=4$; then $2v=K$, and $\operatorname{cn} 2v=0$; hence $t=2$, and

$$m^2=\frac{4(t-1)^2}{t}=2;$$

therefore the curve $\quad (xy+zx-yz)^2=2xyz^2$

is simultaneously inscribed in and circumscribed about an infinite series of quadrilaterals.

Again, let $n=5$; then $5v=2K$ or $4K$, so that

$$\operatorname{cn} 2v \pm \operatorname{cn} 3v = 0.$$

Hence, from (4) and (5),

$$(2-t)(5t-4)\pm t(2t-1)(5-4t)=0;$$

which, after dividing by $t-1$ and $t+1$, gives

$$8t^2-17t+8=0, \quad 8t^2-11t+8=0.$$

Each of these equations gives only one value of m, for

$$m^2=4\left(t+\frac{1}{t}-2\right);$$

thus $\qquad m^2=\frac{1}{2} \quad \text{or} \quad -\frac{8}{2}.$

Hence, for the pentagon, the curves may be written

$$2(xy+zx-yz)^2- xyz^2 = 0,$$

$$2(xy+zx+yz)^2-5xyz^2 = 0,$$

where, in the second case, we have changed the sign of y.

Let $n = 6$; then $6v = 2K$, $3v = K$ and $\operatorname{cn} 3v = 0$, so that (5) gives $t = \frac{5}{4}$, and $m^2 = \frac{1}{5}$; thus the curve is

$$5(xy+zx-yz)^2-xyz^2 = 0.$$

Let $n = 7$; then $7v = 2K$ or $4K$, and $\operatorname{cn} 3v \pm \operatorname{cn} 4v = 0$.

Hence, from (5) and (6), we get

$$\frac{5-4t}{4-5t} \mp \frac{16t^3-33t^2+24t-8}{t(8t^3-24t^2+33t-16)} = 0;$$

which, after dividing by $t-1$ and $t+1$, gives

$$32t^4-37t^3+14t^2-37t+32 = 0,$$

$$32t^4-184t^3+297t^2-184t+32 = 0;$$

or, putting $$t+\frac{1}{t} = \lambda,$$

$$32\lambda^2-37\lambda-50 = 0, \quad 32\lambda^2-184\lambda+233 = 0.$$

Hence, since from these equations there are four real values of

$$\lambda = 2+\tfrac{1}{4}m^2,$$

we infer that there are four curves corresponding to the figure of seven sides.

Let $n = 8$; then $8v = 2K$, $4v = K$, and $\operatorname{cn} 4v = 0$; whence, from (6),

$$16t^3-33t^2+24t-8 = 0.$$

This equation has only one real root, as its Hessian quadratic $7t^2-40t+22$ has both its factors real. There is thus but one real curve in the case of the figure of eight sides.

Let $n = 9$; then $9v = 2K$. Let $3a = 2K$, and $a = 3v$. Now it is easy to show that, if $3a = 2K$ or $4K$, then

$$\operatorname{cn} a = (1 \pm \operatorname{cn} a)\operatorname{dn} a.$$

Hence, from (5), we have

$$(5-4t)(5t-4)\sqrt{(1+t^2-t)} = 9(t-1)(4t^2-7t+4),$$

or $$(5-4t)(5t-4)\sqrt{(1+t^2-t)} = (t+1)(4t^2-7t+4),$$

which, if we put $$t+\frac{1}{t}=\lambda,$$

become $$(20\lambda-41)^2(\lambda-1)-81(\lambda-2)(4\lambda-7)^2=0,$$

$$(20\lambda-41)^2(\lambda-1)-(\lambda+2)(4\lambda-7)^2=0,$$

so that there are six curves for $n=9$.

Let $n=10$; then $10v=2K$, $5v=K$, $3v=K-2v$, and

$$\operatorname{sn}3v=\operatorname{sn}(K-2v)=\frac{\operatorname{cn}2v}{\operatorname{dn}2v};$$

therefore $$9t^2(t^2-1)=(2-t)^2(5t-4)^2,$$

which gives four curves for $n=10$.

A few additional remarks on the curve corresponding to the quadrilateral may be of interest. Putting $m^2=2$, and $\theta\sqrt{2}$ for θ, we may represent the curve thus

$$x=2\theta^2(2\theta^2+2\theta-1),\quad y=2\theta^2+2\theta-1,\quad z=2\theta^2,$$

in which case, if the tangent at θ meet the curve again at θ_1, θ_2, we have

$$\theta_1+\theta_2+2\theta=-1,\quad \frac{1}{\theta_1}+\frac{1}{\theta_2}+\frac{2}{\theta}=2,$$

whence $$\theta_1^2+\theta_2^2+\theta_1+\theta_2-2\theta_1\theta_2(\theta_1+\theta_2+2)=0.$$

Now this relation may be written

$$\phi(\theta_1)+\phi(\theta_2)=0,$$

where $$\phi(\theta)=\frac{\theta(1+\theta)}{2\theta-1};$$

and the form of the equation proves the existence of an infinite number of quadrilaterals; for the four equations

$$\phi(\theta_1)+\phi(\theta_2)=0,\quad \phi(\theta_2)+\phi(\theta_3)=0,\quad \phi(\theta_3)+\phi(\theta_4)=0,$$
$$\phi(\theta_4)+\phi(\theta_1)=0,$$

are manifestly equivalent to only three independent equations.

Let a, b, c, d be the parameters of the vertices; then, if λ is a parameter, we may evidently write

$$\left.\begin{aligned} a^2+a&=\lambda(2a-1),\quad b^2+b=-\lambda(2b-1)\\ c^2+c&=\lambda(2c-1),\quad d^2+d=-\lambda(2d-1)\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots(a),$$

from the first and third of which we get

$$a+c+1=2\lambda, \quad \text{and} \quad b+d+1=-2\lambda;$$

similarly from the second and last. Hence

$$a+b+c+d=-2.$$

Now, if θ is the point of contact of ab,

$$2\theta=-(1+a+b), \quad \text{and} \quad 2\theta'=-(1+c+d),$$

where θ' is the point of contact of cd. Therefore $\theta+\theta'=0$, which, it is easy to see, shows that θ, θ' are collinear with the node. Thus we see that the line joining the points of contact of opposite sides passes through the node; or we may put it: the node is the intersection of the diagonals of the quadrangle formed by the points of contact.

Again, if the curve be reciprocated, it will be transformed into a similar curve, and the property of the quadrilateral remains unaltered. Hence, from the last result, we deduce that the tangents at opposite vertices of the quadrilateral intersect on the bitangent of the curve.

We now prove a property of this curve. The points where the nodal tangents meet the curve again are found by altering the sign of θ in the quadratic giving the parameters of the node, that is

$$2\theta^2+2\theta-1=0,$$

so that the nodal tangents cut the curve again in

$$2\theta^2-2\theta-1=0;$$

but these are the points of inflexion; for the latter points, we have

$$3\theta+\theta'=-1, \quad \frac{3}{\theta}+\frac{1}{\theta'}=2,$$

whence, eliminating θ', we get

$$2\theta^2-2\theta-1=0.$$

This property is the reciprocal of the result noticed by Professor Cayley in my former paper, namely, that the cuspidal tangents intersect the curve again in the points of contact of the bitangent. From the first and third of the equations (a), we easily get $\lambda=ac$, and from the second and fourth $\lambda=-bd$, from which we further deduce

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}=4, \quad ac+bd=0.$$

And from these we propose to prove a result concerning a certain conic circumscribing the quadrilateral. If any conic meet the curve, we can easily show that

$$\Sigma_8^1 \theta = -2, \quad \Sigma_8^1 \frac{1}{\theta} = 4,$$

for the eight points of intersection. Hence, if the conic pass through the node whose parameters are given by

$$2\theta^2 + 2\theta - 1 = 0,$$

we have for the remaining six points of intersection

$$\Sigma_6^1 \theta = -1, \quad \Sigma_6^1 \frac{1}{\theta} = 2.$$

Hence, if the conic passing through a, b, c, d meet the curve again in θ_1, θ_2, we have from the above

$$\theta_1 + \theta_2 - 2 = -1;$$

or
$$\theta_1 + \theta_2 = 1, \quad \frac{1}{\theta_1} + \frac{1}{\theta_2} + 4 = 2,$$

or
$$\frac{1}{\theta_1} + \frac{1}{\theta_2} = -2;$$

but these equations, as we have shown already, determine the points of inflexion. Thus we have proved that we can describe a conic through the vertices of any one of the quadrilaterals, so as to pass through the node and the two points of inflexion.

Furthermore, we can show that this conic passes through another fixed point; for the nodal tangents being

$$x^2 + y^2 - 4xy = 0,$$

and the line of inflexion $\quad 4z + y - x = 0,$

a conic circumscribing that triangle may be written

$$x^2 + y^2 - 4xy - (lx + my)(4z + y - x) = 0,$$

which intersects the curve in points given by

$$(2\theta^2 + 2\theta - 1)^3 (2\theta^2 - 2\theta - 1)$$
$$- (2l\theta^2 + m)(2\theta^2 + 2\theta - 1)(2\theta^3 - 2\theta - 1)(1 - 4\theta - 2\theta^2) = 0;$$

or, dividing out by factors,

$$(2\theta^2 + 2\theta - 1)^2 - (2l\theta^2 + m)(1 - 4\theta - 2\theta^2) = 0,$$

which becomes

$$4(l+1)(\theta^4+2\theta^3)+2(m-l)\theta^2+(m-1)(4\theta-1)=0,$$

of which a, b, c, d are to be the roots.

Now we have $$ac+bd=0,$$

which, in the case of $\theta^4-p\theta^3+q\theta^2-r\theta+s=0$,

gives $$p^2+\frac{r^2}{s}=4q,$$

and, applied to the biquadratic just found, gives, since $p=-2$,

and $$\frac{r}{s}=4, \quad 1+r=q.$$

Thus $$\frac{m-l}{2(l+1)}=1+\frac{1-m}{l+1},$$

or $$m-l=\tfrac{4}{3},$$

which seems to show that the conic passes through the additional point

$$x+y=0, \quad 8z-5y=0.$$

Reciprocating this result, we deduce that a conic can be described to touch the four sides of the quadrilateral, the double tangent, the two cuspidal tangents and another fixed line.

Interesting particular cases might be found by projecting the curve. Thus, if the points of inflexion became the circular points at infinity, the vertices of the quadrilaterals would lie on a system of coaxial circles. Also, if the points of contact of the bitangent were the circular points, the circles circumscribing the four triangles formed by the sides of the quadrilateral would all pass through a fixed point.

We might investigate the relation connecting points whose tangents intersect on the curve as follows. Let the tangent at θ meet the curve again in α, β, then

$$2\theta+\alpha+1=-\beta, \quad \frac{2}{\theta}+\frac{1}{\alpha}-2=-\frac{1}{\beta};$$

therefore, eliminating β, we get

$$2\theta^2(1-2\alpha)+\theta(1+2\alpha-2\alpha^2)+2\alpha(\alpha+1)=0.$$

Now the roots of this equation in θ must be the points of contact of

the tangents which intersect in a. Hence we get

$$\theta_1+\theta_2=\frac{2a^2-2a-1}{2(1-2a)},\quad \theta_1\theta_2=\frac{a(a+1)}{1-2a},$$

so that, eliminating a, we obtain, as the relation between θ_1, θ_2,

$$1+4\theta_1^2\theta_2^2-4(\theta_1^2+\theta_2^2)=0.$$

Now, from the curve, we have

$$y_1/x_1=2\theta_1^2,\quad y_2/x_2=2\theta_2^2.$$

Hence $$x_1x_2+y_1y_2-2(x_1y_2+y_1x_2)=0,$$

which shows that the two points are conjugate with regard to

$$x^2+y^2-4xy=0,$$

i.e., the nodal tangents. Reciprocating this result, we find that tangents at the vertices of a quadrilateral divide the bitangent harmonically. Hence if the curve be projected so that the points of contact of the bitangent become the circular points, the tangents at the extremities of a chord which touches the curve cut at right angles; and the tangents at the vertices of the simultaneously inscribed and circumscribed quadrilateral form a rectangle.

Thursday, April 14th, 1892.

Prof. GREENHILL, F.R.S., President, in the Chair.

Mr. Arthur Lee Dixon, B.A., Fellow of Merton College, Oxford, was elected a member.

The meeting then balloted for the following six mathematicians—Messrs. Poincaré, Hertz, Mittag-Leffler, Schwarz, Beltrami, and Willard Gibbs, whose names had been submitted by the Council for the compliment of foreign membership. All were unanimously elected.

The following communications were made:—

Second Note on a Quaternary Group of 51840 Linear Substitutions: Dr. G. G. Morrice.

Note on the Skew Surfaces applicable upon a given Skew Surface: Prof. Cayley.

Mr. Kempe made an impromptu communication on "Regular Graphs."

The following presents were received:—

"Royal Society—Proceedings" Vol. L., Nos. 305, 306.

"Beiblätter zu den Annalen der Physik und Chemie," Bd. XVI., St. 3; Leipzig, 1892.

"Gesells. der Wissenschaften zu Leipzig—Berichte," Pt. IV., 1892.

"Gesells. der Wissenschaften zu Leipzig—Abhandlungen," Band XVIII., Nos. 3, 4.

"Bulletin of New York Mathematical Society," Vol. I., No. 1, October, 1891, and No. 6, March, 1892. From Prof. W. W. Johnson.

"Kjobenhavn Tidsskrift for Mathematik," A. Tredje Aargang, Nos. 1 and 2.

"Kjobenhavn Tidsskrift for Mathematik," B. Tredje Aargang, No. 1.

"Bulletin de la Société Mathématique de France," Tome XX., No. 1.

"Bulletin des Sciences Mathématiques," Tome XVI., Fév., 1892.

"Roma—Atti della Reale Accademia dei Lincei," Vol. I., Sem. 1, Fasc. 3, 4, 5.

"Educational Times" for April.

"Toulouse—Annales de la Faculté des Sciences," Tome VI., Fasc. 1; Paris, 1892.

"Milano—Annali di Matematica," Tome XIX., Fasc. 4; February, 1892.

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Second Note on a Quaternary Group of 51840 *Linear Substitutions.*

By Dr. G. G. Morrice. Received and Read April 14th, 1892.

In considering complex numbers of the form

$$a_0\rho_0 + a_1\rho_1 + a_2\rho_2 + \&c.,$$

we may attribute to the symbols ρ_0, ρ_1, ρ_2, &c. any significance we please, provided that they conform to the proper multiplication-table. They may be steps along lines, rotations, strains, substitutions, or what not.

In the matter now under consideration they are quaternary matrices.

Let us start from a quaternion $d+ai+bj+ck$; to multiply this by a second quaternion $d'+a'i+b'j+c'k$ is to subject the parameters c, b, a, d to the matrix

$$\begin{pmatrix} d', & a', & -b', & c' \\ -a', & d', & c', & b' \\ b', & -c', & d', & a' \\ -c', & -b', & -a', & d' \end{pmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

viz., we produce the quaternion

$$d''+a''i+b''j+c''k,$$

where

$$\begin{aligned} a'' &= ad'+a'd-(bc'-b'c), \\ b'' &= bd'+b'd-(ca'-c'a), \\ c'' &= cd'+c'd-(ab'-a'b), \\ d'' &= -aa'-bb'-cc'+dd'. \end{aligned}$$

The matrix (1) may be exhibited as a linear function of the matrices

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \ldots\ldots\ldots\ldots(2),$$

as appears at once if we multiply these severally by the scalar parameters a, b, c, d and add. Moreover these four matrices form a group, and indeed we might regard the symbols i, j, k, 1 as being nothing else than symbols for these matrices. That a quaternion is a binary matrix has long been recognised, but it appears to me that its connection with quaternary matrices is even more obvious, and has a better claim to notice, because without it the composition formulæ

$$a'' = ad' + a'd - (bc' - b'c), \text{ \&c.}$$

present no definite idea to our minds.

The process of exhibiting a matrix as a linear function of matrices of special forms occurs in kinematics. A strain is split up into the sum of a uniform dilatation, a skew strain, and a wry shear. The fact that the components form a group is not emphasized.

I now recur to the note on this subject which I had the honour of reading to this Society on December 12th, 1889.

We have four functions z_1, z_2, z_3, z_4 connected with the multiplication by 3 of the normal periods of the double theta-functions, viz.,

$$z_1 = X_{01} - X_{02},$$
$$z_2 = X_{10} - X_{20},$$
$$z_3 = X_{11} - X_{22},$$
$$z_4 = X_{12} - X_{21},$$

where $X_{\alpha\beta}(v_1, v_2; \tau_{11}, \tau_{12}, \tau_{22})$

$$= p_{12}^{\frac{1}{2}(k-1)} \cdot e^{i\pi/k \cdot \phi(\alpha,\beta) + (2\alpha v_1 + 2\beta v_2) i\pi} \cdot \vartheta \begin{pmatrix} kv_1 + \alpha\tau_{11} + \beta\tau_{12} \\ kv_2 + \alpha\tau_{12} + \beta\tau_{22} \end{pmatrix} ; k\tau_{11}, k\tau_{12}, k\tau_{22}$$

$$\times \frac{e^{ki\pi \cdot \Phi(v_1, v_2)}}{\vartheta(\tau_{11}, \tau_{12}, \tau_{22})^k}.$$

I found a sub-group of 4 linear substitutions of the periods ω, all of which were also to be found in the z-group, *i.e.*, the group of 51840 linear substitutions which the functions z_1, z_2, z_3, z_4 undergo; but I

did not notice that the simplicity of the group was accounted for by its isomorphism with quaternions. In fact my ω sub-group is exactly (2).

Following Heinrich Burkhardt, "Untersuchungen aus dem Gebiete der Hyperelliptischen Modulfunctionen," *Math. Annalen*, XXXVIII., 2, our matrices (2) are

$$B^2D, \quad (BD)^3B^2, \quad BDB, \quad 1,$$

to which we have, as corresponding matrices in the z-group,

$$\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 1 \\ -1 & 1 & 0 & -1 \\ 1 & 1 & -1 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & -1 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

If then we multiply by the scalar parameters $a, b, c, d,$ and add, we find that when the periods ω are subjected to the matrix

$$\begin{pmatrix} d, & a, & -b, & c \\ -a, & d, & c, & b \\ b, & -c, & d, & a \\ -c, & -b, & -a, & d \end{pmatrix},$$

the functions z_1, z_2, z_3, z_4 are subjected to the matrix

$$\begin{pmatrix} d-b, & c-a, & -b+c, & b+c \\ c+a, & d-b, & b+c, & b-c \\ -b+c, & b+c, & d-c, & -b-a \\ b+c, & b-c, & -b+a, & d+c \end{pmatrix}.$$

We should have expected to arrive at a matrix of the same form, but with the signs of the four elements in the top corner on the right-

hand side reversed, viz.,

$$\begin{vmatrix} d-b, & c-a, & b-c, & -b-c \\ c+a, & d+b, & -b-c, & -b+c \\ -b+c, & b+c, & d-c, & -b-a \\ b+c, & b-c, & -b+a, & d+c \end{vmatrix} \quad \ldots\ldots\ldots\ldots\ldots (3),$$

for we can easily verify that

$$\begin{vmatrix} d-b, & \ldots, & \ldots, & \ldots \\ \ldots, & \ldots, & \ldots, & \ldots \\ \ldots, & \ldots, & \ldots, & \ldots \\ \ldots, & \ldots, & \ldots, & \ldots \end{vmatrix} \times \begin{vmatrix} d'-b', & \ldots, & \ldots, & \ldots \\ \ldots, & \ldots, & \ldots, & \ldots \\ \ldots, & \ldots, & \ldots, & \ldots \\ \ldots, & \ldots, & \ldots, & \ldots \end{vmatrix}$$

$$= \begin{vmatrix} d''-b'', & \ldots, & \ldots, & \ldots \\ \ldots, & \ldots, & \ldots, & \ldots \\ \ldots, & \ldots, & \ldots, & \ldots \\ \ldots, & \ldots, & \ldots, & \ldots \end{vmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4),$$

where $d'' = -aa'-bb'-cc'+dd'$, &c.;

that is: the matrix (3) has the same law of composition as quaternions.

We seem to require a notation for expressing the fact that the matrix (4) arises from the matrix (3) in two distinct ways: either by subjecting the letters c, b, a, d in the matrix to the matrix

$$\begin{vmatrix} d', & a', & -b', & c' \\ -a', & d', & c', & b' \\ b', & -c', & d', & a' \\ -c', & -b', & -a', & d \end{vmatrix},$$

or multiplying (3) in the ordinary way by the matrix

$$\begin{vmatrix} d'-b', & \ldots, & \ldots, & \ldots \\ \ldots, & \ldots, & \ldots, & \ldots \\ \ldots, & \ldots, & \ldots, & \ldots \\ \ldots, & \ldots, & \ldots, & \ldots \end{vmatrix}.$$

It is only an extension to complex numbers generally of what is familiar in the case of vectors, viz., that the linear and vector function of a vector is a ternary matrix, but there should be a notation independent of the representation of matrices by complex numbers.

An example may be cited for binary matrices. Cayley, *Messenger of Mathematics*, Vol. XIV., p. 178, gives, for a binary matrix Q such that

$$qQ - Qq' = 0,$$

the form
$$\begin{pmatrix} -f\xi - g\eta - h\zeta, & b\xi - a\eta + h\omega \\ -c\xi + a\zeta + g\omega, & c\eta - b\zeta + f\omega \end{pmatrix},$$

and we require a notation to show that this is derived by subjecting the elements of the matrix

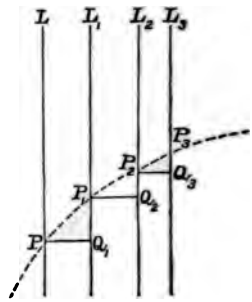
$$\begin{pmatrix} \xi, & \eta \\ \zeta, & \omega \end{pmatrix}$$

to the matrix
$$\begin{pmatrix} -f, & -g, & -h, & 0 \\ b, & -a, & 0, & h \\ -g, & 0, & a, & g \\ 0, & g, & -b, & f \end{pmatrix}.$$

Note on the Skew Surfaces applicable upon a given Skew Surface.

By Prof. CAYLEY. Received March 26th, 1892. Read April 14th, 1892.

The question was considered by Bonnet—§ 7 of his "Mémoire sur la théorie générale des Surfaces," *Jour. Ecole Polyt.*, Cah. 32 (1848); I resume it here, making a greater use of the line of striction.



We may construct a skew surface, inextensible but flexible about its generating lines, as follows: Imagine a flexible extensible plane, and in it the rigid parallel lines L, L_1, L_2, L_3, &c. connected each with the following one by the rigid lines PQ_1, P_1Q_2, P_2Q_3, &c., where PQ_1 cuts L, L_1, P_1Q_2 cuts L_1, L_2, &c. at right angles; the angles LPP_1, $L_1P_1P_2$, $L_2P_2P_3$, &c. are taken to be ω, ω_1, ω_2, &c. respectively. Keeping the line L fixed, we may twist the whole plane $L_1L_2L_3\ldots$ round PQ_1, so that the line L_1 becomes inclined at a small angle to L, these lines now having PQ_1 for their shortest distance, and the lines L_2, L_3, &c. remaining parallel to L_1 in its new position; the foregoing twisting implies an extension (increasing with the distance on each side from P) of the strip or element between the lines L, L_1; after this twisting we imagine the strip in question to become rigid. The amount of the twist is such that if $d\phi$ be the inclination of the lines L, L_1, we have $d\phi = PQ_1 \div r$, viz., the twist $d\phi$ is in a certain proportion to the shortest distance PQ_1. Similarly, keeping the line L_1 fixed, we twist the whole plane $L_2L_3\ldots$ round P_1Q_2, the amount of the twist or inclination $d\phi$ of the lines L_1 and L_2 being $= P_1Q_2 \div r_1$, and after the twist we imagine the strip or element between the two lines L_1, L_2 to become rigid. Proceeding in this manner, we have the series of rigid elements LL_1, L_1L_2, L_2L_3, &c.; putting the line L in any given position, and then keeping it fixed, we may turn the element LL_1 round this line so as to bring L_1 into a certain position; then, keeping L_1 fixed in this position, we may turn the element L_1L_2 round L_1 so as to bring L_2 into a certain position, and so on. To further explain this, imagine through a point O a series of lines K, K_1, K_2, K_3, &c. such that the inclination of K and K_1 is equal to that of L and L_1, the inclination of K_1 and K_2 is equal to that of L_1 and L_2, and so on, but which are otherwise arbitrary; or say that we have the double-triangle strips or elements KK_1, K_1K_2, K_2K_3, &c., bounded by pairs of lines, at given infinitesimal inclinations the two lines of a pair to each other, and forming a flexible double pyramid, which may be bent into any given form whatever assumed at pleasure; and this being so, we see that the system of the strips or elements LL_1, L_1L_2, L_2L_3, &c. may be so bent that the lines L, L_1, L_2, &c. shall be parallel to the lines K, K_1, K_2, &c. respectively.

Supposing the distances PQ_1, P_1Q_2, P_2Q_3, &c. to be all of them

infinitesimal, we have a skew surface containing upon it a curve $P_1P_2P_3$, &c., which is the line of striction, viz., this is the locus of the point on a generating line which is the nearest point to the consecutive generating line. The line of striction cuts the several generating lines at an angle ω, variable from line to line, which is called the obliquity; and the inclination between two consecutive generating lines is in a certain ratio to the shortest distance between the two lines. Inclination = shortest distance $\div r$, this magnitude r being variable from line to line, and its reciprocal r^{-1} is called the "torsion": the obliquity ω and the reciprocal of the torsion r may be regarded as functions of s the arc of the line of striction measured from any fixed point. The skew surface is thus composed of rigid strips or elements, each included between two consecutive lines; and we have further seen that the surface may be bent by turning these rigid elements about the successive generating lines, in such wise that the generating lines become parallel to the generating lines of an arbitrary cone, which is called the "asymptotic" cone (otherwise the director cone); say the surface may be bent so that it shall have a given asymptotic cone.

I consider a given skew surface; I take x, y, z for the coordinates of a point on the line of striction, and α, β, γ for the cosine-inclinations of the generating line through this point; x, y, z, α, β, γ are regarded as functions of s, the length of or distance along the line of striction measured from any fixed point thereof, and I use accents to denote differentiation in regard to s. We have

$$\alpha^2 + \beta^2 + \gamma^2 = 1,$$

and therefore

$$\alpha\alpha' + \beta\beta' + \gamma\gamma' = 0;$$

also

$$x'^2 + y'^2 + z'^2 = 1;$$

and therefore

$$x'x'' + y'y'' + z'z'' = 0;$$

x', y', z' are the cosine-inclinations of the tangent at the point x, y, z.

I remark also that, writing

$$\rho^{-1} = \sqrt{x''^2 + y''^2 + z''^2}$$

(so that ρ is the radius of absolute curvature), we have $\rho x''$, $\rho y''$, $\rho z''$ for the cosine inclinations of the binormal (or perpendicular to the osculating plane).

If from the point (x, y, z) we draw a perpendicular to the consecutive generating line through the point $(x+dx, y+dy, z+dz)$, this will

also be perpendicular to the generating line through the point (x, y, z), and we thus find

$$\alpha'x'+\beta'y'+\gamma'z'=0$$

as the condition that the point (x, y, z) may be, as it is assumed to be, a point on the line of striction.

For the proof hereof, take for the moment x_1, y_1, z_1 the coordinates of P, and $\alpha_1, \beta_1, \gamma_1$ for the cosine-inclinations of L; then, considering the line PQ_1, this passes through P, and cuts the line L_1; taking its equation to be

$$\frac{X-x}{A}=\frac{Y-y}{B}=\frac{Z-z}{C},$$

this meets the line $\dfrac{X-x_1}{\alpha_1}=\dfrac{Y-y_1}{\beta_1}=\dfrac{Z-z_1}{C}$,

and we thence find $\begin{vmatrix} A, & B, & C \\ x_1-x, & y_1-y, & z_1-z \\ \alpha_1, & \beta_1, & \gamma_1 \end{vmatrix}=0.$

Writing $x_1, y_1, z_1 = x+x'ds, y+y'ds, z+z'ds$, ds divides out, and we have

$$A(\beta_1z'-\gamma_1y')+B(\gamma_1x'-\alpha_1z')+C(\alpha_1y'-\beta_1x')=0;$$

the line in question cuts L_1 and L; that is, we have

$$A\alpha_1+B\beta_1+C\gamma_1=0,$$

$$A\alpha+B\beta+C\gamma=0;$$

or, eliminating A, B, C, we find

$$(\beta\gamma_1-\beta_1\gamma)(\beta_1z'-\gamma_1y')+(\gamma\alpha_1-\gamma_1\alpha)(\gamma_1x'-\alpha_1z')+(\alpha\beta_1-\alpha_1\beta)(\alpha_1y'-\beta_1x')=0,$$

that is

$$(\alpha_1x'+\beta_1y'+\gamma_1z')(\alpha\alpha_1+\beta\beta_1+\gamma\gamma_1)-(\alpha x'+\beta y'+\gamma z')(\alpha_1^2+\beta_1^2+\gamma_1^2)=0.$$

But we have $\alpha_1^2+\beta_1^2+\gamma_1^2=1,$

and, writing $\alpha_1, \beta_1, \gamma_1 = \alpha+\alpha'ds, \beta+\beta'ds, \gamma+\gamma'ds$, we find

$$\alpha\alpha_1+\beta\beta_1+\gamma\gamma_1=\alpha^2+\beta^2+\gamma^2+(\alpha\alpha'+\beta\beta'+\gamma\gamma')ds=1;$$

the equation thus is

$$(\alpha_1x'+\beta_1y'+\gamma_1z')-(\alpha x'+\beta y'+\gamma z')=0;$$

that is $$\alpha'x'+\beta'y'+\gamma'z'=0,$$

the required equation.

Calling the inclination of the generating line through the point (x, y, z) to the tangent of the line of striction the "obliquity," and denoting it by ω, we have

$$\alpha x'+\beta y'+\gamma z'=\cos\omega.$$

Calling the inclination of the two consecutive generating lines divided by the shortest distance between these lines the "torsion," and denoting it by τ^{-1}, we have

$$\alpha'^2+\beta'^2+\gamma'^2=\frac{\sin^2\omega}{\tau^2}.$$

In proof hereof, if for a moment ϕ_1 is the inclination of the lines L, L_1 to each other, then

$$\cos\phi=\alpha\alpha_1+\beta\beta_1+\gamma\gamma_1;$$

and therefore $\sin^2\phi=(\beta\gamma_1-\beta_1\gamma)^2+(\gamma\alpha_1-\gamma_1\alpha)^2+(\alpha\beta_1-\alpha_1\beta)^2$,

viz., writing $\alpha_1, \beta_1, \gamma_1 = \alpha+\alpha' ds, \beta+\beta' ds, \gamma+\gamma' ds$, this is

$$\begin{aligned}\sin^2\phi &= ds^2\left\{(\beta\gamma'-\beta'\gamma)^2+(\gamma\alpha'-\gamma'\alpha)^2+(\alpha\beta'-\alpha'\beta)^2\right\}\\ &= ds^2\left\{(\alpha^2+\beta^2+\gamma^2)(\alpha'^2+\beta'^2+\gamma'^2)-(\alpha\alpha'+\beta\beta'+\gamma\gamma')^2\right\},\\ &= ds^2(\alpha'^2+\beta'^2+\gamma'^2);\end{aligned}$$

whence, if for a moment the shortest distance between the two lines is called δ, then we have

$$\frac{\sin\phi}{\delta}=\tau^{-1}=\frac{\sin\phi}{ds\sin\omega};$$

that is $$\frac{\sin^2\omega}{\tau^2}=\frac{\sin^2\phi}{ds^2}=\alpha'^2+\beta'^2+\gamma'^2,$$

the required equation.

We have thus the five equations

$$\begin{aligned}\alpha^2+\beta^2+\gamma^2&=1,\\ x^2+y^2+z^2&=1,\\ \alpha'x'+\beta'y'+\gamma'z'&=0,\\ \alpha x'+\beta y'+\gamma z'&=\cos\omega,\\ \alpha'^2+\beta'^2+\gamma'^2&=\frac{\sin^2\omega}{\tau^2},\end{aligned}$$

and if we herein consider ω and r as denoting given functions of s, all the skew surfaces which satisfy these equations will be surfaces applicable one on the other.

Adding to the foregoing the derived equations

$$\alpha\alpha' + \beta\beta' + \gamma\gamma' = 0,$$
$$x'x'' + y'y'' + z'z'' = 0,$$
$$\alpha x'' + \beta y'' + \gamma z'' = -\sin\omega \cdot \omega',$$
$$\alpha x''' + \beta y''' + \gamma z''' + \alpha' x'' + \beta' y'' + \gamma' z'' = -\sin\omega \cdot \omega'' - \cos\omega \cdot \omega'^2,$$

we find without difficulty

$$\alpha', \beta', \gamma' = \frac{\beta z' - \gamma y'}{r}, \frac{\gamma x' - \alpha z'}{r}, \frac{\alpha y' - \beta x'}{r},$$

$$\beta\gamma' - \beta'\gamma, \ \gamma\alpha' - \gamma'\alpha, \ \alpha\beta' - \alpha'\beta$$
$$= \frac{-x' + x\cos\omega}{r}, \frac{-y' + \beta\cos\omega}{r}, \frac{-z' + \gamma\cos\omega}{r}.$$

Putting, for shortness,

$$\Theta = -\sin\omega \cdot \omega'' - \cos\omega \cdot \omega'^2 - (\alpha x''' + \beta y''' + \gamma z'''),$$

$$\nabla = \begin{vmatrix} \alpha, & \beta, & \gamma \\ x', & y', & z' \\ x'', & y'', & z'' \end{vmatrix},$$

and, as above, $\rho^{-2} = x''^2 + y''^2 + z''^2$,

we find $\nabla^2 = \sin^2\omega \left(\frac{1}{\rho^2} - \omega'^2\right)$, $r = \frac{\nabla}{\Theta} = \frac{\sin\omega}{\rho\Theta}\sqrt{1 - \rho^2\omega'^2}$,

and to these may be joined

$$\beta z' - \gamma y' = \rho^2 \{\nabla x'' + \omega' \sin\omega \, (y'z'' - y''z')\},$$
$$\gamma x' - \alpha z' = \rho^2 \{\nabla y'' + \omega' \sin\omega \, (z'x'' - z''x')\},$$
$$\alpha y' - \beta x' = \rho^2 \{\nabla z'' + \omega' \sin\omega \, (x'y'' - x''y')\}.$$

I remark that, supposing the line of striction to be given (that is, (x, y, z) to be given as functions of s), and moreover the obliquity ω to be given as a function of s, the position of the generating line

through the point (x, y, z) will be given, and also the torsion τ^{-1}. In fact, among the foregoing equations, we have

$$\alpha^2 + \beta^2 + \gamma^2 = 1,$$

$$\alpha x' + \beta y' + \gamma z' = \cos\omega,$$

$$\alpha x'' + \beta y'' + \gamma z'' = -\sin\omega \cdot \omega,$$

which equations determine α, β, γ, that is, the position of the generating line; the position of the consecutive generating line is then also determined; we thus have α', β', γ', and thence the torsion, which is given by the foregoing equation

$$\tau = \frac{\sin\omega}{\rho\Theta}\sqrt{1-\rho^2\omega^2}.$$

Supposing that the line of striction is not given, but that the obliquity ω and the torsion τ^{-1} are given as functions of s, we have then the foregoing *five* equations, which are not sufficient for the determination of $x, y, z, \alpha, \beta, \gamma$; but, joining to them an assumed homogeneous relation between (α, β, γ), the six quantities will be determined. The assumed homogeneous relation between (α, β, γ) is the equation of the asymptotic cone; and we have thus the theorem that, given the asymptotic cone, and also the obliquity and the torsion as functions of s (the arc of the line of striction), then this line of striction, and the skew surface the locus of the generating lines, will be determined.

We have between α, β, γ the assumed homogeneous equation, say $U = 0$; and among the foregoing equations the equations

$$\alpha^2+\beta^2+\gamma^2 = 1, \quad \text{and} \quad \alpha'^2+\beta'^2+\gamma'^2 = \frac{\sin^2\omega}{\tau^2},$$

a given function of s; these equations give, by means of an integration, α, β, γ as functions of s; and we have then the above-mentioned equations

$$x' - \alpha\cos\omega,\ y' - \beta\cos\omega,\ z' - \gamma\cos\omega = \tau(\beta\gamma' - \beta'\gamma,\ \gamma\alpha' - \gamma'\alpha,\ \alpha\beta' - \alpha'\beta),$$

which give, by integration, x, y, z as functions of s. And the skew surfaces thus obtained for an assumed asymptotic cone $U = 0$ will be the required system of skew surfaces applicable the one on the other.

For example, if ω, τ are constants, and we take, for $U = 0$, the equation

$$\gamma^2 - c(\alpha^2+\beta^2) = 0$$

(that is, if we consider the skew surfaces of constant obliquity and torsion and which have for asymptotic cone a right circular cone), it easily appears that the line of striction is a helix traced upon a right circular cylinder, and that the generating lines are at a constant inclination to the axis of the cylinder, and all of them touch the cylinder. In fact, for such a surface (a kind of helicoid) it is in the first place obvious that the helix is the line of striction, and next that the obliquity and the torsion are each of them constant.

Supposing that the obliquity and the torsion are ω, τ, and that the inclination of the generating lines to the axis of the cylinder is $90° - \omega - \delta$, δ being a given constant, we find

$$x = \frac{\tau \cos\delta \cos(\omega+\delta)}{\sin\omega} \cos\left(\frac{s\sin\omega}{\tau\cos(\omega+\delta)}\right),$$

$$y = \frac{\tau \cos\delta \cos(\omega+\delta)}{\sin\omega} \sin\left(\frac{s\sin\omega}{\tau\cos(\omega+\delta)}\right),$$

$$z = s\sin\delta;$$

$$\alpha = -\cos(\omega+\delta) \quad \sin\left(\frac{s\sin\omega}{\tau\cos(\omega+\delta)}\right),$$

$$\beta = \cos(\omega+\delta) \quad \cos\left(\frac{s\sin\omega}{\tau\cos(\omega+\delta)}\right),$$

$$\gamma = \sin(\omega+\delta);$$

where observe that the line of striction lies on the circular cylinder,

$$\text{radius} = \tau\cos\delta\cos(\omega+\delta) \div \sin\omega.$$

In particular, if $\delta = 0$, we have

$$x = \tau\cot\omega\cos\frac{s}{\tau\cot\omega}, \quad \alpha = -\cos\omega\sin\frac{s}{\tau\cot\omega},$$

$$y = \tau\cot\omega\sin\frac{s}{\tau\cot\omega}, \quad \beta = \cos\omega\cos\frac{s}{\tau\cot\omega},$$

$$z = 0, \qquad \gamma = \sin\omega,$$

and, for $\delta = -\omega$, we have

$$x = \tau\cot\omega\cos\frac{s\sin\omega}{\tau}, \quad \alpha = -\sin\frac{s\sin\omega}{\tau},$$

$$y = \tau\cot\omega\sin\frac{s\sin\omega}{\tau}, \quad \beta = \cos\frac{s\sin\omega}{\tau},$$

$$z = -s\sin\omega, \qquad \gamma = 0.$$

The first of these is the skew hyperboloid of revolution

$$X^2+Y^2-(Z^2+r^2)\cot^2\omega = 0$$

(radius of gorge $= r\cot\omega$), and the second of them is the helicoid (radius of cylinder $= r\cot\omega$) the generating lines of which are the tangents perpendicular to the axis of the cylinder; these are in fact surfaces found by Bonnet in his memoir above referred to. We may imagine the surface passing from the first form, in which the semi-aperture of the asymptotic cone is ω, through the series of forms belonging to the general formulæ involving δ, to the second form, in which the semi-aperture is $= 90°$, *i.e.* in which the asymptotic cone is replaced by a plane.

Thursday, May 12th, 1892.

Prof. GREENHILL, F.R.S., President, in the Chair.

The following gentlemen were elected members of the Society:—Messrs. G. T. Bennett, B.A., Scholar of St. John's College, Cambridge, and W. E. Heal, of Indiana, U.S.A., and Dr. F. A. Tarleton, F.T.C.D., Professor of Natural Philosophy in the University of Dublin.

Mr. A. E. Jolliffe was admitted into the Society.

The following communications were made:—

On an Operator that produces all the Covariants and Invariants of any System of Quantics: Dr. W. E. Story.

Applications of a Theory of Permutations in Circular Procession to the Theory of Numbers: Major P. A. Macmahon, R.A.

A Newtonian Fragment relating to Centripetal Forces: Mr. W. W. Rouse Ball.

The following presents were received:—

"Beiblätter zu den Annalen der Physik und Chemie," Band XVI., Stück 4; Leipzig, 1892.

"Jornal de Sciencias Mathematicas e Astronomicas," Vol. X., No. 5; Coimbra, 1892.

"Berichte über die Verhandlungen der K. S. Gesells. der Wiss. zu Leipzig," 1891, V.

"Physical Society of London—Proceedings," Vol. xi., Pt. 3, March, 1892.

"Physical Society of London—List of Officers and Members," April 1st, 1892.

"Bulletin de la Societé Mathématique de France," Tome xx., No. 2; Paris, 1892.

"Bulletin of the New York Mathematical Society," Vol. i., Nos. 1–7; New York, 1892.

"Bulletin des Sciences Mathématiques," Tome xvi., Mars, 1892; Paris, 1892.

"Rendiconti del Circolo Matematico di Palermo," Tomo vi., Fasc. 1, 2, 1892.

"Educational Times," May, 1892.

"Annals of Mathematics," Vol. vi., No. 5; University of Virginia, 1892.

"Treatise on Physical Optics," A. B. Basset (London, George Bell, 1892). From the Author.

"Beiträge zur Geometrie des Dreiecks," Dr. Jos. Hahn. From the Author.

"On Bessel's Functions of the Second Kind," Dr. M. Bôcher. From the Author.

"Factor Tables," Lieut.-Col. W. H. Oakes. ("Annals of Mathematics.")

A Newtonian Fragment relating to Centripetal Forces.

By W. W. Rouse Ball. Read May 12th, 1892.

1. The demonstrations given by Newton in his *Principia* are geometrical, though there is little doubt that in establishing the truth of some of his results he used fluxions.* To his contemporaries the language and methods of geometry were familiar, while to most of them the calculus was unknown; hence it was natural and reasonable that the proofs should be presented in a geometrical form. I presume that the fluxional analysis by which a result was obtained was generally thrown aside as soon as a synthetic geometrical proof had been found; and, as far as I know, the only proposition in that book of which Newton's fluxional demonstration has been published is his determination of the form of the solid of least resistance, of which the result alone was given in the *Principia*.†

2. Among the numerous sheets of rough work and calculation which

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* *Commercium Epistolicum*, edition of 1722, p. 39; S. P. Rigaud, ***Essay on the First Publication of the Principia***, Oxford, 1838, p. 24; Sir David Brewster, ***Life of Newton***, second edition, Edinburgh, 1860, vol. i., p. 347.

† *Principia*, book ii. scholium to prop. 35 in the first edition, and to prop. 34 in the second and third editions.

are preserved in the Portsmouth Collection is a fragment* on the law of the centripetal force under which any orbit—and particularly a parabola of any order—can be described. The theorem to which the analysis leads is so inconvenient of application as to be practically useless, and, probably for that reason, it was not inserted in the *Principia*. Such interest as it possesses lies rather in its illustrating the way in which Newton arrived at the law given below for the description of any parabola under a central force.

The fragment consists merely of a double sheet of folio paper, containing calculations and rough drafts of notes subsequently incorporated in the second edition; but there are so many erasures and corrections that, were it not for Newton's proverbially clear handwriting, it would be difficult to read the manuscript.

3. The two inside pages contain (i.) a note on the velocity of sound from which the second paragraph added in the second edition to the scholium to book II. prop. 50 is taken, and (ii.) numerical calculations showing that the moon is retained in her orbit by the earth's attraction. The latter calculations refer explicitly to book III. prop. 4 as printed in the first edition, where the mean distance of the moon in syzygy is taken as 61 times the earth's radius; but he here says that more accurate calculations show that the mean distance of the moon is 59·9 (or 59·8) semi-diameters of the earth if the earth be at rest, and is $60\frac{1}{3}$ (or $60\frac{1}{4}$) semi-diameters if the earth and moon revolve about their common centre of gravity; in the second edition the distance is taken as being approximately equal to $60\frac{1}{2}$ radii of the earth: though in both editions he uses the number 60 as giving a sufficiently close approximation.

Of the two outside pages, the first begins with the draft of a scholium — presumably to book I. prop. 7 — which is entitled "Methodus investigandi vires quibus corpora in Orbibus propositis revolventur amplior reddi potest per Propositiones sequentes," and then follow three rules, of which the last is incomplete. Book I. prop. 7 is on the law of force under which a circle can be described about a centre of force S in its own plane. In the first edition, S is a point on the circumference, and there are no corollaries. In the second edition, S is any point in the plane; cor. 1 relates to the case when S is on the circumference; cor. 2 is concerned with the comparison of the forces to two different points about which the circle

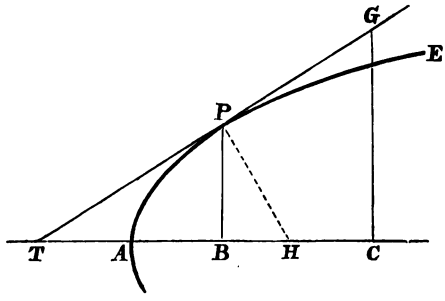
* *Portsmouth Collection*, I. VIII. 2.

might be described; and in cor. 3, the result of cor. 2 is extended to any orbit. The first rule, given here by Newton, is on similar orbits described about centres of force similarly situated. The second rule and the third (so far as it is written) seem to be the same, and are given in a form substantially identical with the third corollary, which was added to prop. 7 in the second edition. On the next page Newton refers to these rules, and says that therefore "per Prop. 10 et Cor. 3, Prop. 7" the law of force under which a central conic can be described about any point is known, because (by prop. 10) the law to the centre is known. This is the alternative proof for book I. props. 11 and 12, which was introduced in the second edition, and is printed in most modern text-books. The above rules enable us to find the law of force to any point under which a given orbit can be described, provided the law to one point is known.

4. Newton next returns to the problem (already treated in prop. 6) of the direct determination of the centripetal force to any given point under which any given orbit can be described. This is the proposition to which I referred above as illustrative of his use of fluxions. I quote the manuscript textually, merely adding punctuation. Perhaps it may be convenient to prefix the remark that the square of a quantity x is generally denoted by x^q (for *x quadratus*), and the cube is denoted by x^{cub}.

"Si Lex vis centripetae investiganda sit qua corpus P in orbe quocunque APE circa centrum quodcunque datum C movebitur: institui potest calculus per methodum sequentem.

"Sit AB Abscissa Curvae propositae, per centrum C transiens; sitque BP ejus Ordinata in dato quovis angulo Abscissae insistens. Ducatur TPG Orbem tangens in P, et Abscissae occurrens in T; et



Ordinatae BP parallela agatur CG tangenti occurrens in G. Ffluat Abscissa uniformiter, et exponatur ejus fluxio per unitatem. Et si

Ordinata BP dicatur v, vis centripeta qua corpus P in Orbe APE circa centrum C movebitur erit ut

$$-\frac{CP \times \ddot{v}}{CG^{cub}}.$$

5. Newton then applies this theorem to determine the centripetal force under which any parabola can be described. In this application, H is the foot of the normal drawn through P, and the axes are assumed to be rectangular. The manuscript continues as follows.

"Ut si aequatio ad Curvam sit $ax = v^n$ (ubi a quantitatem quamvis datum et x Abscissam denotat, et n index est dignitatis Ordinatae v). Methodus fluxionum dabit primo

$$a = n\dot{v}v^{n-1},$$

deinde,
$$0 = (nn-n)\dot{v}\dot{v}v^{n-2} + n\ddot{v}v^{n-1},$$

seu
$$\ddot{v} = -\frac{(n-1)\dot{v}\dot{v}}{v},$$

id est,
$$= -\frac{(n-1) \times PB}{TB^q}.$$

Et propterea vis centripeta erit ut

$$\frac{(n-1) \times PB \times CP}{TB^q \times CG^{cub}},$$

id est (ob datum $n-1$, et aequalia $TB \times CG$ et $PB \times TC$) ut

$$\frac{CP}{TB \times TC \times CG^q},$$

vel quod perinde est ut

$$\frac{CP}{PB \times TC^q \times CG}$$

$$= \frac{CP}{BH \times TC^{cub}}."$$

6. Newton also applies the method to the curve whose equation is $ax^m + bx^n = cy^p$. He finds $\ddot{y}$, but the expression for the force thence derived is not simple or symmetrical, and he appears to have thrown the theorem aside as not being useful.

7. No demonstration of the theorem quoted above in art. 4 is given

in the manuscript; but, if the axes be rectangular, the result follows at once from book I. prop. 6. For, with the usual notation, taking (see figure, art. 4) $AB = x$, $BP = v$, and denoting CP by r, the perpendicular from C on TP by p, the angle T by ψ, and the radius of curvature at P by ρ, we have

$$F = \frac{h^2}{p^3}\frac{dp}{dr}$$

$$= \frac{h^2}{p^3}\frac{CP}{\rho}$$

$$= \frac{h^2 CP}{\rho CG^3 \cos^3\psi}.$$

Now

$$\frac{1}{\rho\cos^3\psi} = \frac{d\psi}{ds}\frac{ds}{dx}\sec^2\psi$$

$$= \frac{d}{dx}(\tan\psi)$$

$$= \frac{d}{dx}\left(\frac{dv}{dx}\right)$$

$$= \ddot{v}.$$

Hence

$$F \propto \frac{CP\ddot{v}}{CG^3}.$$

8. As first written, Newton stated that the axes must be rectangular, and it would seem that the subsequent alteration to oblique axes was an error. For, if the axes be inclined at an angle ω, an argument similar to that in the last article shows that

$$F = \frac{h^2}{p^3}\frac{dp}{dr} = \frac{h^2}{p^3}\frac{r}{\rho} = \frac{h^2 CP}{\rho CG^3 \sin^3(\omega-\psi)},$$

and

$$\frac{1}{\rho\sin^3(\omega-\psi)} = \frac{d\psi}{ds}\left\{\frac{1}{\sin\omega}\frac{ds}{dx}\right\}\operatorname{cosec}^2(\omega-\psi) = \frac{1}{\sin\omega}\frac{d}{dx}\{\cot(\omega-\psi)\}.$$

Hence, in order that Newton's result may be true, we must have $\cot(\omega-\psi)$ equal to $\frac{dv}{dx}$, that is, to $\frac{\sin\psi}{\sin(\omega-\psi)}$; this is the case only if $\omega = \frac{1}{2}\pi$, that is, if the axes be rectangular.

9 The date of this fragment is not of much importance, but I am inclined to fix it as about the year 1694, when we know that Newton

was engaged in revising the first edition of the *Principia*. The reference to prop. 7 cor. 3 looks at first sight as if the manuscript were subsequent to the second edition; but I think that this reference relates merely to the manuscript of his proposed addition to that proposition, of which the three rules mentioned above are, I believe, a rough draft. On the other hand, the numerical calculation relating to book III. prop. 4 gives a number as occurring on page 406, line 25, and can refer only to the first edition; moreover many of the remarks alluded to above in art. 3 would be meaningless if written subsequent to the publication of the second edition in 1713. Altogether I feel no doubt that the manuscript was written before the issue of the second edition.

The Harmonic Functions for the Elliptic Cone. *By* E. W. Hobson. Communicated in abstract January 14th, 1892. Received May 30th, 1892.

The harmonic functions for the circular cone were introduced by Mehler; an account of his theory is given in Heine's *Kugelfunctionen*. In the present communication, I give some indications of a theory of the more general harmonic functions which are required for the corresponding potential problems connected with the elliptic cone; I propose to call these harmonics *elliptic conal harmonics*.

It is first shown that the normal functions are of the form

$$\frac{1}{\sqrt{r}} \begin{matrix} \sin \\ \cos \end{matrix} (p \log r)\, A_p(\mu)\, B_p(\nu),$$

where r is the radius vector, and μ, ν are elliptic coordinates, the latter referring to the elliptic cones, and p is a constant; the functions $A_p(\mu)$, $B_p(\nu)$ satisfy differential equations which are the same as Lamé's, except that the degree n is no longer a positive integer, but a complex quantity $-\frac{1}{2}+pc$, so that the functions are really Lamé's functions of complex degree. I have next considered the forms of the solutions of the differential equations satisfied by

$A_p(\mu)$, $B_p(\nu)$ which must be taken for the potential problem. There are no solutions, as in Lamé's case, which can be found in finite terms. The constant parameter which occurs in the differential equations is determined from the condition that $A_p(\mu)$ must be periodic with respect to a certain angle ϕ, of which μ is a function; it appears that $A_p(\mu)$ falls into four distinct classes corresponding to Lamé's four classes, and that the constant is determined as a root of one of four transcendental equations. I have next considered the nature of the roots of these equations, and shown that these roots are all real and are contained between certain intervals. It is next shown that two solutions $B_p(\nu)$, $B_p(-\nu)$ of the equation satisfied by B, exist, one of which is finite everywhere within the cone, but infinite over a certain space lying in a principal plane of the cone produced beyond the vertex.

So far as I am aware, Lamé's equation has been hitherto studied only in the case in which n is a positive integer (by Lamé and Hermite), and in the case in which n is half an odd positive integer.* It appeared to me that the indications I have been able to give as to the nature of the solutions in the case of these more complicated functions, might not be without interest.

It may be remarked that the problem of electrical distribution on an infinite plate bounded by two straight edges meeting at a point, is solvable in terms of these functions. Mehler's functions are, of course, a particular case of the functions I have considered.

1. If for the rectangular coordinates x, y, z we put

$$x = r\frac{\mu\nu}{bc}, \quad y = r\frac{\sqrt{\mu^2-b^2}\sqrt{b^2-\nu^2}}{b\sqrt{c^2-b^2}}, \quad z = r\frac{\sqrt{c^2-\mu^2}\sqrt{c^2-\nu^2}}{c\sqrt{c^2-b^2}},$$

where $c>b$, we have $x^2+y^2+z^2 = r^2$,

$$\frac{x^2}{\mu^2}+\frac{y^2}{\mu^2-b^2}-\frac{z^2}{c^2-\mu^2}=0,$$

$$\frac{x^2}{\nu^2}-\frac{y^2}{b^2-\nu^2}-\frac{z^2}{c^2-\nu^2}=0;$$

thus the equations $r =$ constant, $\mu =$ constant, $\nu =$ constant, represent three systems of surfaces cutting each other orthogonally, the

* See Halphen's *Fonctions elliptiques*, Vol. II., p. 482; also Lindemann, *British Association Report*, 1883.

first being concentric spheres, and the second and third two systems of confocal cones having their vertices at the centre of the spheres. We are about to consider the potential problem for spaces in which a boundary is the half-cone for which ν is constant and positive; for the other half of the cone ν will be negative. We observe that ν may have values of 0 to b^2, and μ^2 may have values from b^2 to c^2.

It is known* that Laplace's equation $\nabla^2 V = 0$, when the quantities r, μ, ν are considered as the coordinates of a point, takes the form

$$(\mu^2-\nu^2)\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)+\frac{\partial^2 V}{\partial \epsilon^2}+\frac{\partial^2 V}{\partial \zeta^2}=0,$$

where $$\mu = c\,\mathrm{dn}\,(K-c\epsilon,\, k), \quad \nu = b\,\mathrm{sn}\,(\zeta,\, k'),$$

k being $\frac{1}{c}\sqrt{c^2-b^2}$, and k' the complementary modulus; this equation is satisfied by $(Cr^n+Dr^{-n-1})\,A(\mu)\,B(\nu)$, where $A(\mu)$, $B(\nu)$ satisfy the equations

$$\frac{d^2A(\mu)}{d\epsilon^2}+\{n(n+1)\mu^2-(b^2+c^2)\alpha\}A(\mu)=0,$$

$$\frac{d^2B(\nu)}{d\zeta^2}-\{n(n+1)\nu^2-(b^2+c^2)\alpha\}B(\nu)=0,$$

α being any constant quantity.

If n is determined so that the solution $(Cr^n+Dr^{-n-1})\,A(\mu)\,B(\nu)$ vanishes for two given values r_1 and r_2 of r, we must have

$$Cr_1^n+Dr_1^{-n-1}=0, \quad Cr_2^n+Dr_2^{-n-1}=0;$$

hence $$r_1^{2n+1}=r_2^{2n+1},$$

or $$(2n+1)\log\frac{r_1}{r_2}=2k\iota\pi,$$

where $\iota=\sqrt{-1}$, and k is an integer; hence

$$n=-\tfrac{1}{2}+\iota\frac{k\pi}{\log\frac{r_1}{r_2}}.$$

* See Heine's *Kugelfunctionen*, p. 354.

Thus, as in the case of Mehler's functions, we take n of the form $-\frac{1}{2}+\iota p$, for harmonics adapted to the case of the cone; the normal functions are therefore of the form

$$\frac{1}{\sqrt{r}} \begin{matrix}\sin \\ \cos\end{matrix} p\rho \,.\, A_p(\mu)\, B_p(\nu),$$

when $$\rho = \log r,$$

and where $A_p(\mu)$, $B_p(\nu)$ satisfy the equations

$$\frac{d^2 A_p(\mu)}{d\epsilon^2} - \left[(p^2+\tfrac{1}{4})\mu^2 + (b^2+c^2)\alpha\right] A_p(\mu) = 0 \quad \ldots\ldots\ldots(1),$$

$$\frac{d^2 B_p(\nu)}{d\zeta^2} + \left[(p^2+\tfrac{1}{4})\nu^2 + (b^2+c^2)\alpha\right] B_p(\nu) = 0 \quad \ldots\ldots\ldots(2),$$

which differ from Lamé's equation only in having $-\frac{1}{2}+p\sqrt{-1}$ for n; thus $A_p(\mu)$, $B_p(\nu)$ may be regarded as Lamé's functions of complex degree.

These equations are reducible to

$$(\mu^2-b^2)(\mu^2-c^2)\frac{d^2A(\mu)}{d\mu^2} + \mu(2\mu^2-b^2-c^2)\frac{dA(\mu)}{d\mu} + \left[(b^2+c^2)\alpha + (p^2+\tfrac{1}{4})\mu^2\right] A(\mu) = 0,$$

and a precisely similar equation in ν.

2. In order to determine the forms of $A(\mu)$, we shall put

$$\cos\phi = \sqrt{\frac{\mu^2-b^2}{c^2-b^2}},$$

and we observe that, in order that the normal function may be a potential function either for the space inside the half-cone $\nu =$ constant, or for the space outside this half-cone, it is necessary that $A(\mu)$ should be a periodic function of ϕ, the period being 2π; the values of the constant α will be determined for given values of p by means of a certain convergency condition.

The equation $$\frac{d^2A}{d\epsilon^2} - \left[(p^2+\tfrac{1}{4})\mu^2 + (b^2+c^2)\alpha\right] A = 0 \quad \ldots\ldots\ldots(1)$$

is equivalent, when μ is the independent variable, to the equation

$$(\mu^2-b^2)(\mu^2-c^2)\frac{d^2A}{d\mu^2}+\mu(2\mu^2-b^2-c^2)\frac{dA}{d\mu}$$
$$+\{(b^2+c^2)\alpha+(p^2+\tfrac{1}{4})\mu^2\}A=0.$$

Putting $$\mu=(b^2\sin^2\phi+c^2\cos^2\phi)^{\frac{1}{2}},$$

and making ϕ the independent variable, this equation becomes

$$(b^2\sin^2\phi+c^2\cos^2\phi)\frac{d^2A}{d\phi^2}-(c^2-b^2)\sin\phi\cos\phi\frac{dA}{d\phi}$$
$$-\{(b^2+c^2)\alpha+(p^2+\tfrac{1}{4})(b^2\sin^2\phi+c^2\cos^2\phi)\}A=0,$$

or, writing $$\frac{c^2-b^2}{c^2+b^2}=\kappa,$$

$$(1+\kappa\cos 2\phi)\frac{d^2A}{d\phi^2}-\kappa\sin 2\phi\frac{dA}{d\phi}-\{2\alpha+(p^2+\tfrac{1}{4})(1+\kappa\cos 2\phi)\}A=0.$$

To solve this equation, we may take A of one of the four forms which correspond to Lamé's four classes,

$$\tfrac{1}{2}a_0+a_2\cos 2\phi+a_4\cos 4\phi+\ldots,$$
$$a_2\sin 2\phi+a_4\sin 4\phi+a_6\sin 6\phi+\ldots,$$
$$a_1\cos\phi+a_3\cos 3\phi+a_5\cos 5\phi+\ldots,$$
$$a_1\sin\phi+a_3\sin 3\phi+a_5\sin 5\phi+\ldots.$$

These series are infinite instead of finite as in Lamé's case.

Substituting the first of these series in the differential equation, arranging the result of the substitution in a series of cosines of multiples of ϕ, and equating the coefficient of the general term to zero, we find

$$-4n^2a_{2n}-2(n-1)^2\kappa a_{2n-2}-2(n+1)^2\kappa a_{2n+2}+(n+1)\kappa a_{2n+2}-(n-1)\kappa a_{2n-2}$$
$$-(2\alpha+p^2+\tfrac{1}{4})a_{2n}-(p^2+\tfrac{1}{4})\tfrac{1}{2}\kappa a_{2n-2}-(p^2+\tfrac{1}{4})\tfrac{1}{2}\kappa a_{2n+2}=0,$$

and in particular $$a_0(2\alpha+p^2+\tfrac{1}{4})+\kappa a_2(p^2+\tfrac{1}{4})=0.$$

Arranging the above result, we have for the equation connecting a_{2n+2}, a_{2n}, a_{2n-2},

$$a_{2n+2}\{(p^2+\tfrac{1}{4})\tfrac{1}{2}\kappa+(n+1)(2n+1)\kappa\}+a_{2n}\{4n^2+(2\alpha+p^2+\tfrac{1}{4})\}$$
$$+a_{2n-2}\{(n-1)(2n-1)\kappa+(p^2+\tfrac{1}{4})\}\tfrac{1}{2}\kappa=0.$$

Write $$-\beta = 2a + p^2 + \tfrac{1}{4}, \quad \tfrac{1}{2}\kappa\,(p^2 + \tfrac{1}{4}) = \gamma;$$

then, changing n into $n-1$, we have

$$a_{2n}\{\gamma + n(2n-1)\kappa\} + a_{2n-2}\{4(n-1)^2 - \beta\} + a_{2n-4}\{(n-2)(2n-3)\kappa + \gamma\} = 0 \quad \ldots\ldots\ldots(3)$$

as the equation connecting a_{2n}, a_{2n-2}, a_{2n-4}.

By means of the n equations obtained by giving n the values 1, 2, 3 ... n, we can express a_{2n} in terms of a_0, the value of a_{2n}/a_0 being a rational integral expression in β or in a of degree n. The condition of convergency of the series is that the limit of a_{2n}, when n is indefinitely increased, may be zero; this condition gives the values of β or a as the roots of a transcendental equation which it is necessary to examine.

If any one of the other three series is substituted in the differential equation, we shall obtain in a similar manner a relation connecting the coefficients, and thence a transcendental equation for the determination of β or a, as the convergency condition of the series. It will, however, be sufficient for us to examine the first class of solutions, as the treatment of the other three classes is precisely similar.

3. It will now be shown that all the n roots of the equation $a_{2n} = 0$, in the quantity β, are real and unequal.

Writing down the equations (3) for the values 1, 2, 3, ... n of n, we see that, when $\beta = -\infty$, the functions $a_0, a_2, a_4, \ldots a_{2n}$ have only changes of sign, and when $\beta = +\infty$, they have only continuations; thus, as β changes from ∞ to $-\infty$, n continuations of sign are lost. When $a_{2r} = 0$, where $r < n$, we see that a_{2r+2} and a_{2r-2} have opposite signs; hence, as in the case of Sturm's functions, a continuation of signs is lost only when a_{2n} goes through the value zero; it follows that the number of roots of the equation $a_{2n} = 0$ in β, between any two given values of β, is equal to the number of continuations of sign in the series $a_0, a_2, a_4, \ldots a_{2n}$ which are lost as β goes from the larger of the two values to the smaller one, and that all the roots are real and unequal.

Again, if $\beta = 4(n-1)^2$, we see that a_{2n} and a_{2n-4} have opposite signs; hence $a_{2n} = 0$ has one or more roots lying between ∞ and $4(n-1)^2$.

Next reduce $\frac{a_{2n}}{a_{2n-2}}$ to a continued fraction of the form

$$\beta-4(n-1)^2-\frac{1}{u_n\{\beta-4(n-2)^2\}}-\frac{1}{v_{n-1}\{\beta-4(n-3)^2\}}-\&c.,$$

where $u_n=\frac{n(2n-1)\kappa+\gamma}{(n-2)(2n-3)\kappa+\gamma}$, $v_{n-1}=\frac{u_{n-1}}{u_n}$, &c.

If we put $\beta=4n^2$, it can be seen that each denominator in the continued fraction is greater than 2, and thus that the whole expression is positive. The same is the case if β has any value greater than $4n^2$; hence for the value $\beta=4n^2$ the signs of a_{2n}, a_{2n-2}, a_{2n-4}, ... are all the same, and thus there is no root of the equation $a_{2n}=0$ greater than $4n^2$. When $\beta=4(n-1)^2$, the series a_{2n-2}, a_{2n-4} ... are all of the same sign, and thus there is only one change in the series a_{2n}, a_{2n-2}, a_{2n-4}, ...; and therefore the equation $a_{2n}=0$ has one and only one root lying between the values $4n^2$ and $4(n-1)^2$ of β; this root is the greatest one which the equation has.

We shall next show that the equation $a_{2n}=0$ has r roots less than $4r^2$. In the series a_0, a_2, ... a_{2r}, there are only continuations when $\beta=4r^2$; hence in the series a_0, a_2, ... a_{2n} there are $n-r$ changes; thus there are $n-r$ roots of $a_{2n}=0$ greater than $4r^2$, and therefore r roots less than $4r^2$. It has thus been shown that, of the roots of the equation $a_{2n}=0$, one lies between each of the numbers $0, 2^2, 4^2, \ldots (2n)^2$. The roots of $a_\infty=0$ are therefore comprised between $0, 2^2, 4^2, \ldots$, one root being in each interval.

In the special case $\kappa=0$, we have $b=c$, and the differential equation reduces to

$$\frac{d^2A}{d\phi^2}+\beta A=0;$$

in this case the values of β are $0, 2^2, 4^2, \ldots$, and the corresponding values of the functions of this class are 0, $a_2\cos 2\phi$, $a_4\cos 4\phi$, $a_6\cos 6\phi$, ...; these are the functions for the circular cone.

In the case of the functions A of the second class, it is shown in a similar manner that there are an infinite number of values of β, corresponding to any one of which a function is determined, and that these values of β lie between the same intervals as in the case of the first class.

In the functions of the third and fourth classes, the values of β lie in the intervals between $0, 1^2, 3^2, 5^2, \ldots$.

In the case of the circular cone ($\kappa = 0$), the four classes of functions A are $\cos 2n\phi$, $\sin 2n\phi$, $\cos(2n+1)\phi$, $\sin(2n+1)\phi$.

4. We have next to consider the form of the function $B_p(\nu)$, α having a value corresponding to a particular function $A_p(\mu)$ as above determined.

We have to integrate the equation

$$(\nu^2-b^2)(\nu^2-c^2)\frac{d^2B(\nu)}{d\nu^2}+\nu(2\nu^2-b^2-c^2)\frac{dB(\nu)}{d\nu}$$
$$+\left[(b^2+c^2)\alpha+(p^2+\tfrac{1}{4})\nu^2\right]B(\nu)=0,$$

in a form suitable for values of ν numerically less than b. The singular points being $\nu=\pm b$, $\nu=\pm c$, the equation has an integral of the form

$$A_0\left\{1+a_2\left(\frac{\nu}{b}\right)^2+a_4\left(\frac{\nu}{b}\right)^4+\ldots\right\}$$
$$+B_0\frac{\nu}{b}\left\{1+b_2\left(\frac{\nu}{b}\right)^2+b_4\left(\frac{\nu}{b}\right)^4+\ldots\right\},$$

which is convergent when $\nu<b$, A_0, B_0 denoting arbitrary constants, and a_2, a_4, ..., b_2, b_4, ... quantities which can be determined. When $\nu=\pm b$, each of the series becomes divergent, but it is possible to choose the value of A_0/B_0 so that the whole expression will be convergent for $\nu=b$; for this value of A_0/B_0, the expression will be infinite when $\nu=-b$. Taking for A_0/B_0 the same value as before, but with opposite sign, we obtain a solution which is convergent for $\nu=-b$ but divergent for $\nu=b$. Denoting this value of A_0/B_0 by $1/\lambda$, we see that the equation has two solutions

$$A_0\left\{\left(1+a_2\frac{\nu^2}{b^2}+a_4\frac{\nu^4}{b^4}+\ldots\right)+\lambda\frac{\nu}{b}\left(1+b_2\frac{\nu^2}{b^2}+b_4\frac{\nu^4}{b^4}+\ldots\right)\right\},$$

$$A_0\left\{\left(1+a_2\frac{\nu^2}{b^2}+a_4\frac{\nu^4}{b^4}+\ldots\right)-\lambda\frac{\nu}{b}\left(1+b_2\frac{\nu^2}{b^2}+b_4\frac{\nu^4}{b^4}+\ldots\right)\right\},$$

the first of which is finite when $\nu=b$, and is infinite when $\nu=-b$, and the second of which is finite when $\nu=-b$, and infinite when $\nu=b$. If we denote the expression

$$\left(1+a_2\frac{\nu^2}{b^2}+a_4\frac{\nu^4}{b^4}+\ldots\right)+\lambda\left(1+b_2\frac{\nu^2}{b^2}+b_4\frac{\nu^4}{b^4}+\ldots\right)$$

by $B_p(\nu)$, the complete solution of the equation is

$$B(\nu) = CB_p(\nu) + C'B_p(-\nu).$$

The function $B_p(\nu)$ is finite throughout the space which contains the infinite triangle $\nu = b$, and is infinite in the external space which contains the infinite triangle $\nu = -b$. There exists one such function $B_p(\nu)$ corresponding to each function $A_p(\mu)$. The constant λ is a transcendental quantity to be determined; in the case ($b = c$) of the circular cone, its value can be determined.*

In this last case the functions $B_p(\nu)$, $B_p(-\nu)$ become Mehler's functions $K(\cos\theta)$, $K(-\cos\theta)$, the first of which is finite for $\theta = 0$, and the second for $\theta = \pi$.

5. As in the case of Lamé's functions, it can be shown that, if $A_{p_1}(\mu)$, $A_{p_2}(\mu)$ be two functions corresponding to a given value of p, and both belonging to the same one of the four classes, then

$$\int_0^\infty A_{p_1} A_{p_2} d\epsilon = 0.$$

To expand a function of μ in a series of the functions A, we divide the function into four parts

$$f_1(\mu) + \sqrt{\mu^2-b^2}\sqrt{c^2-\mu^2}\, f_2(\mu) + \sqrt{\mu^2-b^2}\, f_3(\mu) + \sqrt{c^2-\mu^2}\, f_4(\mu),$$

corresponding to the four classes of A functions, where $f_1(\mu)$, $f_2(\mu)$, $f_3(\mu)$, $f_4(\mu)$ are rational functions of μ; then, assuming the possibility of the expansions, the coefficients in the expansion may be determined as in the case of Lamé's functions.

6. Suppose it is required to find a potential function, inside the space bounded by the semi-cone $\nu = \nu_0$, which shall have a given value over the surface of this boundary. Let

$$V = \frac{1}{\sqrt{r}} f(r, \mu)$$

be the given value of V when $\nu = \nu_0$; then

$$\frac{1}{\sqrt{r}} f(r, \mu) = \frac{1}{2\pi\sqrt{r}} \int_{-\infty}^{\infty} dp \int_0^\infty f(\sigma, \mu) \cos p(r-\sigma)\, d\sigma$$

* See *Camb. Phil. Trans.*, Vol. XIV., p. 218.

for all values of r between 0 and ∞. Next, suppose $f(\sigma, \mu)$ expanded in a series of A functions corresponding to the value p, say $\sum_{r=1}^{\infty} a_r A_{pr}$; we have then

$$\frac{1}{\sqrt{r}} f(r, \mu) = \frac{1}{2\sqrt{\pi r}} \int_{-\infty}^{\infty} dp \int_0^{\infty} \Sigma a_r A_{pr} \cos p (r-\sigma)\, d\sigma;$$

the required value of the potential function within $\nu = \nu_0$ is

$$\frac{1}{2\pi\sqrt{r}} \int_{-\infty}^{\infty} dp \int_0^{\infty} \Sigma a_r \frac{B_{pr}(\nu)}{B_{pr}(\nu_0)} A_{pr} \cos p (r-\sigma)\, d\sigma.$$

The potential function for the space outside the boundary $\nu = \nu_0$, which has the same value as before over the boundary, is

$$\frac{1}{2\pi\sqrt{r}} \int_{-\infty}^{\infty} dp \int_0^{\infty} \Sigma a_r \frac{B_{pr}(-\nu)}{B_{pr}(-\nu_0)} A_{pr} \cos p (r-\sigma)\, d\sigma.$$

Next consider the space bounded by two spheres $r = a$, $r = b$, and by the conal surface $\nu = \nu_0$; suppose the potential function V is to have the value zero over each of the spherical boundaries and to have prescribed values over the conal boundaries. In this case the values of p are $\frac{k\pi}{\log \frac{a}{b}}$, where k is an integer.

The value of V must be of the form

$$\sum_{k=1}^{k=\infty} \frac{1}{\sqrt{r}} \sin\left\{\frac{k\pi \log \frac{r}{b}}{\log \frac{a}{b}}\right\} \sum_{s=1}^{s=\infty} \alpha A_{k\pi/(\log a/b),\, s}(\mu) \frac{B(\nu)}{B(\nu_0)},$$

the constants α being determined from the assigned values of V over the conal surface $\nu = \nu_0$.

Researches in the Calculus of Variations.—The Discrimination of Maxima and Minima Values of Single Integrals with any Number of Dependent Variables and any continuous Restrictions on the Variables, the Limiting Values of the Variables being supposed given. By EDWARD P. CULVERWELL, M.A., Fellow of Trinity College, Dublin. Received July 21st, 1892. Abstracts were communicated November 12th, 1891, December 10th, 1891, and January 14th, 1892.

The subject of the discrimination of maxima and minima values in the calculus of variations has possessed considerable interest for mathematicians, although, as yet, at all events, of little if any practical utility. But, except for the simplest case, that of

$$\int f\left(x, y, \frac{dy}{dx}\right) dx,$$

the investigations supplied have been so difficult, and have required such an amount of analytical skill, as to deter any but the most competent mathematicians from considering cases of any degree of complexity; and, even for single integrals, the analytic method of solution has not, so far as I am aware, been completed when the variations of the unknown functions are restricted, or when a *function* of different integrals has to be made a maximum. Again, since the analytic solution, as originated by Jacobi, and extended by some later writers, requires us to employ $\frac{d^{2n}\delta y}{dx^{2n}}$ $\left(\text{when } \frac{d^n y}{dx^n}\right.$ is the highest fluxion of y in the integral$\left.\right)$, it would seem only to justify us in stating that the integral is a maximum when compared with others derived from it by variations in which $\frac{d^{2n}\delta y}{dx^{2n}}$ is never infinite. For instance, in the problem of shortest distance between two points P and Q, a variation which leads us along the zig-zag line joining R and Q would not be admissible, and the Jacobian investigation does not tell us whether the straight line is a minimum compared with such lines or not, but only with lines which have a continuous change in the direction of the tangent. This is evidently a defect in the method of investigation, which is further defective inasmuch as it

does not show satisfactorily that when the Jacobian conditions are not satisfied, the maximum or minimum property does not hold, though this is, in fact, the case.

The simplicity of the Jacobian result seemed to indicate that there must be some simple way of obtaining it, and, guided by the desire to ascertain the degree of continuity required in the variation, the simple, almost geometrical, method explained in the following pages presented itself to me, first with respect to the unrestricted problem, and then for the more complex cases in which the variations are restricted. It appears a remarkable circumstance that in every case, no matter how complicated, the criteria follow immediately from the condition that the variations must be such as to render valid the expansions used in obtaining the *stationary* value of U (that for which $\delta U = 0$).

The paper is divided into four parts. In all it is supposed that there is but one *independent* variable, though there may be many *dependent* variables. In Part I., the known results for a single integral U are rigorously established. (Essentially this is a portion of a paper in the *Transactions of the Royal Society*, Vol. CLXXVIII., p. 95, but so modified as to render it easily applicable to the class of problem in Part II.) Part II. deals with what are known as Isoperimetrical Problems, or Relative Maxima and Minima; Part III. with cases in which there are relations among the variables in U; and Part IV. with problems involving *functions* of two or more integrals. It is pointed out at the conclusion that there is no further theoretic difficulty when the peculiarities in Parts II., III., and IV. are combined in one problem.

The case in which the limits are variable is reserved for another communication.

Part I.

1. *Definitions.* (*a*)—$x', x'', U, u, \dot{y}, \ddot{y}, \ldots y^{(n)}, z$.

The limits of the integration are represented by x' and x''; the integral which is to be made a maximum is denoted by

$$U, \text{ or by } \int_{x'}^{x''} u\,dx, \text{ or by } \int_{x'}^{x''} f(x, y, \dot{y}, \ldots y^{(n)})\,dx,$$

according to convenience. The fluxional notation,

$$\dot{y} \equiv dy/dx, \quad \ddot{y} \equiv d^2y/dx^2, \quad y^{(n)} \equiv d^ny/dx^n,$$

being used throughout. Where there is a second dependent variable, it is denoted by z.

(*b*)—δy *and* $a\phi$, U_0, U_1, U_2, ..., U_n; U_{00}, U_{10}, ... U_{nn}.

I find it convenient to use $a\phi$ for δy in many cases, a being a small constant, the order of magnitude of which determines, generally speaking, the order of magnitude of δy, since the function ϕ is supposed to be finite, at least never infinite.

When, in u or $f(x, y, \dot{y}, \ldots y^{(n)})$, the value $y+\delta y$ or $y+a\phi$ is substituted for y, and therefore $\dot{y}+a\dot{\phi}$ for $\dot{y}$, and so on, the result, when expanded in powers of ϕ, $\dot{\phi}$, &c., may be written

$$\left.\begin{aligned} f(x, y+a\phi, \dot{y}+a\dot{\phi}, \ldots y^{(n)}+a\phi^{(n)}) &= f(x, y, \dot{y}, \ldots y^{(n)}) \\ &+ a(U_0\phi + U_1\dot{\phi} + \&c. + U_n\phi^{(n)}) \\ &+ \tfrac{1}{2}a^2(U_{00}\phi^2 + 2U_{01}\phi\dot{\phi} + U_{11}\dot{\phi}^2 + \&c. + U_{nn}\phi^{(n)2}) \\ &+ \text{terms involving } a^3 \text{ and higher powers} \end{aligned}\right\} \ldots (1).$$

This expansion defines the coefficients U_0, &c., U_{00}, &c.

(*c*)—ΔU; δU; $\delta^2 U$; *first variation*; *second variation.*

By ΔU, I denote all the terms involving a in the *integral* of the right-hand side of (1), so that ΔU is the complete difference between the values of U when $y+a\phi$ and y are successively taken as dependent variables.

The term in ΔU which involves a in the *first* degree is called the *first variation of* U, and is denoted by δU; that involving a^2 is the second variation, and may be written $\frac{1}{2}\delta^2 U$ (strictly speaking, this applies only when the limits of the x integration are fixed).

(*d*)—*Stationary value of* U; *stationary curve*; "*fixed limits*"; *conjugate points.*

When the form of y is such that δU vanishes independently of the value of δy, the corresponding value of U may be called the stationary value, the value of y being called its stationary value, or more conveniently the stationary curve, since the expression for y in terms of x may be represented by a curve.

The limits of the dependent variable are said to be given when the values of $y, \dot{y}, \ldots y^{(n-1)}$ are given for each limit of integration, an

this is shortly expressed by saying that "the limits are fixed." *In this case we must have* $\phi, \dot{\phi}, \ldots \phi^{(n-1)}$ *zero at each limit, for otherwise* δy *and its fluxions would not vanish.*

When it is possible to draw between two points P and Q two *consecutive* stationary curves having at P and Q the same values for y and its $(n-1)$ first fluxions, the two points may be called *conjugate*. In this sense a pair of kinetic foci are *conjugate* points.

(*e*)—*Sign of Substitution.*

This is occasionally used, thus $\Big|_{x'}^{x''} U_1$ means $(U_1'' - U_1')$.

2. *Limitations implied in using the expansion by Taylor's Theorem.*

(*a*) The quantities $\phi, \dot{\phi}, \ldots \phi^{(n)}$ which appear in ΔU must remain finite or zero for every value of x included in the integration.

(*b*) The coefficients $U_0, U_1, \ldots U_{00}, U_{10}, \ldots U_{nn}$, must similarly remain finite while y has any value between y and $y + a\phi$.

(*c*) Conditions (*a*) and (*b*) being satisfied, a can be and must be taken small enough to secure that the terms involving a^3 and higher powers do not affect the *sign* of ΔU. It is not, in general, necessary to take a indefinitely small, though it is always sufficient to do so.

These are the only conditions in reference to the magnitude of δy, or $a\phi$, a remark which seems called for in consequence of a criticism made by a competent mathematician, that the method on p. 259 is not admissible, since the variation selected is an infinitely small one of the *second* order, since $h^2 = \rho r$, while variations of the first order only are considered by the calculus of variations. But, in fact, the smaller δy is, the more certain we are that the third condition is fulfilled.

The whole theory of the discrimination of maxima and minima solutions can be derived from considering the continuity of the variation δy, or $a\phi$, as involved in (*a*).

3. Prop. I.—*The function* ϕ *and all its fluxions up to and including* $\phi^{(n-1)}$ *must be in every sense continuous, but* $\phi^{(n)}$ *may be discontinuous in the sense of changing suddenly from one finite value to another.*

The first part of this is evident, because if say $\phi^{(n-1)}$ become discontinuous, $\phi^{(n)}$ would be infinite, contrary to (*a*), § 2. The second

part is evident when we consider the meaning of such a discontinuity in $\phi^{(n)}$. Thus, if

$$U = \int \sqrt{1+\dot{y}^2}\, dx,$$

such a discontinuity would mean that we compare the straight line with an adjacent curve having a sudden change of direction.

PROP. II.—*When the range of integration is small and the "limits are fixed," § 1, the value of $\phi^{(n)}$ at any point is large compared to the value of $\phi^{(n-1)}$ or any lower fluxion.*

Since the limits are fixed, the values of $\phi, \dot{\phi}, \ldots \phi^{(n-1)}$ are zero at each limit, so that, at a point x,

$$\phi_x^{(n-1)} = \int_{x'}^{x} \phi^{(n)}\, dx.$$

Hence, if β be the numerically greatest value of $\phi^{(n)}$ between x' and x'', it follows that $\phi^{(n-1)}$ can never exceed $(x''-x')\beta$, and therefore, by taking $(x''-x')$ sufficiently small, the ratio of $\phi^{(n-1)}$ to $\phi^{(n)}$ may be decreased indefinitely.*

A similar argument shows that *a fortiori* the lower functions may be neglected in comparison with $\phi^{(n)}$.

COROLLARY.—*Hence in this case U is always a maximum or a minimum for fixed limits, according as U_{nn} is negative or positive.*

For the most important term in $\delta^2 U$ is $\int_{x'}^{x''} U_{nn}\phi^{(n)2}\, dx$, and its sign is always the same as that of U_{nn}, dx being of course taken positively.

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* At first sight it might appear impossible that the statement in this proposition could be true, for we determine the stationary value of y by the condition that

$$\delta U \text{ or } a\int_{x'}^{x''} (U_0\phi + U_1\dot{\phi} + \&c. + U_n\phi^{(n)})\, dx$$

shall vanish, so that the different terms must cancel one against the other, and $\phi^{(n)}$ must be comparable to $\phi^{(n-1)}$, &c. But this relates not to the magnitudes of the *different values* of $\phi^{(n)}$, $\phi^{(n-1)}$, &c., but to the magnitudes of *integrals* of these quantities multiplied by functions U_n, $U_{(n-1)}$, &c., which, *in the small range of integration*, are nearly constant; hence $\int_{x'}^{x''} U_n\phi^{(n)}\, dx$ is nearly the same as $U_n\int_{x'}^{x''}\phi^{(n)}\, dx$, which vanishes, because $\phi^{(n-1)}$ vanishes at each limit. In other words, $\phi^{(n)}$ has different signs in different parts of the integration, and these parts mutually destroy each other, so far as terms of the order $\phi^{(n)}$ are concerned. But where $\phi^{(n)}$ enters by its *square*, as in $\int U_{nn}\phi^{(n)2}\, dx$, no such reduction occurs, and all the terms involving lower fluxions may be neglected in comparison with this term.

PROP. III.—*A variation, beginning at x_1 and ending at x_2, from a curve $y = \theta x$ to another $y = \theta(x) + a\phi(x)$, is admissible, provided that $\phi, \dot{\phi}, \ldots \phi^{(n-1)}$ are zero at x_1 and x_2, the beginning and end of the variation.*

For evidently we may suppose ϕ to be zero from x' to x_1 and from x_2 to x'', and to have a value ϕ satisfying the above conditions between x_1 and x_2, and this satisfies the requirements of Taylor's Theorem (of course $\phi^{(n)}$ changes suddenly at x_1 and x_2).

Similarly we may use a variation made up of distinct curves, provided that wherever two of the curves join, they have contact of the $(n-2)^{\text{th}}$ order, *i.e.*, that the values of $y^{(n-1)}$ are the same for both curves.

4. PROP. IV.—*The integral U cannot be a maximum unless $U_{nn}\,dx$ remain negative throughout the whole range of the integration.*

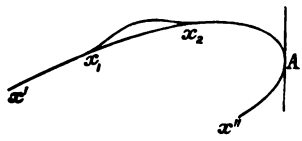

FIG. 1.

Writing the value of $\delta^2 U$ in the form

$$2\delta^2 U = a^2 \int_{x'}^{x''} (U_{00}\phi^2 + \&c. + 2U_{(n-1)n}\phi^{(n-1)}\phi^{(n)} + U_{nn}\phi^{(n)2})\,dx,$$

let us suppose $U_{nn}dx$ to be positive through a *small* part of the integral, viz., from x_1 to x_2. Then, by taking an admissible variation ϕ, which is zero except between those values of x, it follows, from Prop. II., that the only important term in the variation is

$$a^2 \int_{x_1}^{x_2} U_{nn}\phi^{(n)2}\,dx,$$

and the sign of this is evidently the same as that of $U_{nn}dx$; which proves the proposition.

COROLLARY.—If U is a maximum and the stationary curve has a vertical tangent, as at A in the figure, the sign of U_{nn} must change at that point, since the sign of dx evidently changes there.

PROP. V.—*When the limits are fixed, and when $U_{nn}dx$ is negative throughout the integration, the stationary value of U is a maximum, provided x'' is not beyond the point "conjugate" to x'.*

Suppose we take a variation which has the value ϕ between x' and X, and zero between X and x'', $\phi, \dot{\phi}, \ldots \phi^{(n-1)}$ being of course zero at x' and X. Then we have seen (Prop. II., Cor.) that, if X be very near to x', $\delta^2 U$ is always a negative quantity of the order a^2 compared to the value of U when the limits are x' and X. It is therefore evident that, as we remove X further from x', the value of $\delta^2 U$ cannot become capable of a positive value until after X has passed through the point for which $\delta^2 U$ can *vanish* for a suitably chosen value of δy or $a\phi$.

Suppose $PBQR$ to be the stationary curve, Q being the *first point for which it is possible to draw a varied curve PHQ for which $\delta^2 U$ is zero.*

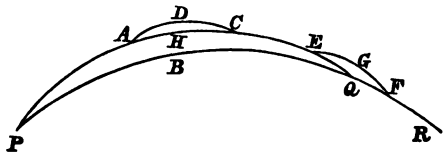

FIG. 2.

Then it is easy to show that, $U_{nn}dx$ being always negative (so that for very close limits of integration the stationary curve is a true maximum) the varied curve PHQ must itself be stationary. For, if it were not stationary between two neighbouring points A and C, suppose a stationary curve ADC to be drawn between those points, such that the variation $PADCQ$ is an admissible one (Prop. IV.), *i.e.*, ADC is to have contact of the $(n-2)^{\text{th}}$ order with PHQ at A and C. Then it would follow, from Prop. II., that the integral taken along ADC would be greater than that taken along the original curve AHC, and therefore the integral along $PADCQ$ would be greater than that along PHQ, which by hypothesis is equal to that along PBQ. Hence $\delta^2 U$ along $PADCQ$ would be *positive*, and therefore Q *could not be the first point for which, as we proceed from P, the variation ceases to be negative.* Thus we see that $PAHCQ$ must be everywhere stationary, so that Q is the point "conjugate" to P (§ 1).

Hence it follows rigorously that U is a true maximum, provided the two conditions hold, first that $U_{nn}dx$ is negative, and second that, in passing along the curve, x'' is reached before the conjugate point.

PROP. VI.—*If $U_{nn}\,dx$ changes sign between x' and x'', or if the point conjugate to x' lies between x' and x'', then U is neither a maximum nor a minimum.*

The first part of this follows at once from Prop. IV. To show the second part, suppose that in Fig. 2 the integration extends from P to R (Q being, as before, the point conjugate to P), and let two points E and F be taken, one on the curve AHQ and the other on QR. Taking these points so close that any stationary curve joining them gives a true maximum, let the stationary curve EGF be drawn, having contact of the order $(n-2)$ with PHQ and QR at E and F respectively; then by Prop. III. the variation $PHEGFR$ is legitimate, and, since the integral along the portion EGF is greater than that along EQF, it follows that the integral taken along $PEGFR$ is greater than that along $PEQR$. But the latter is equal to that taken along $PBQR$, since both integrals are equal from P to Q. Hence $PBQR$ is not a maximum (and it is not a minimum, because the condition for a minimum, viz., that $U_{nn}\,dx$ be everywhere positive, is not fulfilled).

If U_{nn} should vanish without changing sign, we have a curious case, for if we confine ourselves to a variation given in the immediate vicinity of the point for which U_{nn} vanishes, the second variation loses its order $\left(\text{because } \int U_{nn}\phi^{(n)2}dx \text{ vanishes with a vanishing } U_{nn}\right)$, and the third variation asserts itself, so that the total variation ΔU may change sign for this restricted variation. But for any other variation U is a maximum.

5. The equation which determines the conjugate point is well known. If

$$y = f(x, c_1, c_2, c_3, \ldots c_{2n})$$

be the general equation of the stationary curve, and if the partial differential coefficients of y with regard to $c_1, c_2, \ldots c_{2n}$ be written as $y_1, y_2, \ldots y_{2n}$, the δy of a consecutive stationary curve $y+\delta y$ is determined by the equation

$$\delta y = C_1 y_1 + C_2 y_2 + \ldots + C_{2n} y_{2n} \quad \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

where $C_1, C_2, \ldots C_{2n}$ are arbitrary constants; whence, expressing that $\delta y, \delta\dot{y}, \ldots \delta y^{(n-1)}$ all vanish for x' and x'', we obtain $2n$ linear equations

in $C_1, C_2, \ldots C_{2n}$, from which by elimination we obtain the determinant

$$\begin{vmatrix} y'_1, & y'_2, & \ldots\ldots & y'_{2n} \\ \dot{y}'_1, & \dot{y}'_2, & \ldots\ldots & \dot{y}'_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ y_1^{\prime(n-1)}, & y_2^{\prime(n-1)}, & \ldots\ldots & y_{2n}^{\prime(n-1)} \\ y''_1, & y''_2, & \ldots\ldots & y''_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ y_1^{\prime\prime(n-1)}, & y_2^{\prime\prime(n-1)}, & \ldots\ldots & y_{2n}^{\prime\prime(n-1)} \end{vmatrix} = 0 \quad \ldots\ldots\ldots\ldots (3),$$

an equation which gives x'' in terms of x'. The least value of x'', or rather the value first reached in going from x' along the curve, is the conjugate point.

6. If there is more than one dependent variable, a similar method can be applied. Prop. II. applies to each variable, and therefore in Prop. IV. we have only to replace

$$a^2 \int_{x'}^{x''} U_{nn} \phi^{(n)2} dx \quad \text{or} \quad \int_{x'}^{x''} U_{nn} (\delta y^{(n)})^2 dx$$

by an expression which, for the case of two variables y and z, whose n^{th} and m^{th} fluxions respectively appear, may be written in the form

$$\int_{x'}^{x''} \{a (\delta y^{(n)})^2 + 2h (\delta y^{(n)})(\delta z^{(m)}) + b (\delta z^{(m)})^2\} dx.$$

Evidently the condition that this shall be always of the same sign is that a and b shall be of the same sign, and that $ab > h^2$ (which really includes the first condition). If however $ab = h^2$, the terms involving the highest fluxions of the variations can be made to vanish, and it may become necessary to examine the terms with lower fluxions. The problem then assumes a form dealt with in § 10, as is easily seen by writing for $\delta y^{(5)}$ in the $\delta^2 u$ of that paragraph its value $-(b\delta z^{(5)} + c\delta w^{(5)}) \div a$, and treating the problem as one of two variables.

Props. V. and VI. apply without material modification.

Evidently the method is applicable whatever be the number of dependent variables.

Part II. *Isoperimetrical Problems.*

7. Here it is required to make U a maximum, subject to the condition that another integral V is to have a given value K. I use for V a notation exactly similar to that already adopted for U.

According to the ordinary method we have, since $\Delta V \equiv 0$,

$$\Delta U \equiv \Delta U + \lambda \Delta V = \delta U + \lambda \delta V + \tfrac{1}{2} (\delta^2 U + \lambda \delta^2 V) + \&c.$$

The stationary curve makes

$$\delta U + \lambda \delta V \equiv 0,$$

so that, for it,

$$\Delta U = \tfrac{1}{2}(\delta^2 U + \lambda \delta^2 V) + \&c.$$

$$\ldots\ldots = \frac{a^2}{2} \int_{x'}^{x''} \left\{ (U_{00} + \lambda V_{00})\, \phi^2 + \&c. + (U_{nn} + \lambda V_{nn})\, \phi^{(n)2} \right\} dx + \&c.,$$

and, exactly as in Prop. IV., we see that $(U_{nn} + \lambda V_{nn})\, dx$ must be negative throughout the integration if U is a maximum. For evidently the fact that the variations must satisfy the equation $\Delta V = 0$, does not affect the argument in Prop. II.

The argument in Prop. V. is unaffected, except that between x' and X we must of course give such variations that

$$\Delta \int_{x'}^{X} v\, dx = 0,$$

for as

$$\Delta \int_{X}^{x''} v\, dx \equiv 0$$

(because the variation is taken as zero between X and x''), we shall then, and only then, have

$$\Delta \int_{x'}^{X} v\, dx + \Delta \int_{X}^{x''} v\, dx = 0, \text{ or } \Delta \int_{x'}^{x''} v\, dx,$$

that is ΔV, equal zero.

Hence, in this case, we must define the "conjugate point" to x' as the point to which a second stationary curve can be drawn, which shall give the same value to V as the original curve (for which therefore the value of λ will in general be different), and which shall have the same values for $y, \dot{y}, \ldots y^{(n-1)}$ as the original curve at both the limiting points. Hence the equation which determines the conjugate point may be obtained as follows.

Let $y = f(x, \lambda, c_1, c_2, \ldots c_{2n})$ be the equation of the stationary curve, expressed in terms of the arbitrary constants, and let $V = F(x', x'', \lambda, c_1, c_2, \ldots c_{2n})$ be the value of V for the stationary curve when the limits of integration are x' and x''. Then the value of δy given in (2) § 5, must be increased by a term which we may write as $\Lambda \frac{df}{d\lambda}$ or Λy_λ, and we shall have in addition the equation $\Delta V = 0$, which may be written

$$\Delta V = \Lambda F_\lambda + C_1 F_1 + C_2 F_2 + \ldots + C_{2n} F_{2n} = 0,$$

Λ, C_1, C_2, ... C_{2n} being, as in § 5, the variations of the arbitrary constants, and F_λ, F_1, &c. being the partial differential coefficients of F with regard to them. Hence, eliminating, as in § 5, the variations of the constants, we obtain the equation

$$\begin{vmatrix} y_1', & y_2', & \dots\dots & y_{2n}', & y_\lambda' \\ \dot{y}_1', & \dot{y}_2', & \dots\dots & \dot{y}_{2n}', & \dot{y}_\lambda' \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ y_1'^{(n-1)}, & y_2'^{(n-1)}, & \dots\dots & y_{2n}'^{(n-1)}, & y_\lambda'^{(n-1)} \\ F_1, & F_2, & \dots\dots & F_{2n}, & F_\lambda \end{vmatrix} = 0,$$

which determines x'' in terms of x', for the case in which the integral just ceases to give a true maximum.

It is evident that the method may be applied without any material modification when there are several integrals V, W, &c. whose value is given, or again that § 6, in which there are supposed to be several dependent variables, can be modified so as to adapt itself to the case in which there are integrals V, W, &c. involving those variables which have given values in the problem.

PART III. *Integrals with two or more Dependent Variables, connected by Algebraic or Differential Equations.*

8. Let $U = \int_{x'}^{x''} u\,dx$, where u is a function of $y, z, \dot{y}, \dot{z}, \dots y^{(n)}, z^{(n)}$, and let the dependent variables y and z be connected by an equation $v = 0$, v being a function similar to u, and containing the same fluxions.

Since $v = 0$ for all admissible values, we have also $\Delta v = 0$; and therefore ΔU may be written

$$\Delta U = \int_{x'}^{x''} (\Delta u + \lambda \Delta v)\,dx$$

$$= \int_{x'}^{x''} (\delta u + \lambda \delta v)\,dx + \tfrac{1}{2} \int_{x'}^{x''} (\delta^2 u + \lambda \delta^2 v)\,dx + \&c.$$

According to the usual method we determine y, z, and λ, so that the first integral shall vanish, leaving the equation

$$\Delta U = \tfrac{1}{2} \int_{x'}^{x''} (\delta^2 u + \lambda \delta^2 v)\,dx.$$

Using a suffix at the left-hand to denote partial differentiation of u and v with regard to z and its fluxions, just as it has been used at the right-hand with regard to y, we may write, putting $a\theta$ for δz,

$$\delta^2 u = a^2(U_{00}\phi^2 + 2_0U_0\phi\theta + {}_{00}U\theta^2 + \&c. + U_{nn}\phi^{(n)^2} + 2_nU_n\phi^{(n)}\theta^{(n)} + {}_{nn}U\theta^{(n)^2}),$$

and the expression for $\delta^2 v$ will only differ by V taking the place of U.

Now, exactly as in Prop. II., we see that *when the range of integration is small*, $\phi^{(n)}$ is large with respect to $\phi^{(n-1)}$ or lower fluxions of ϕ, and $\theta^{(n)}$ large with respect to $\theta^{(n-1)}$ or lower fluxions of θ. Therefore it follows that when the range of integration is small, we may neglect in $\delta^2 u$ all but the last three terms in the above expression.

Again, ϕ and θ are not independent, but are connected by the equation $\delta v = 0$, in which, if the range of integration be small, we may neglect all variations except $\phi^{(n)}$ and $\theta^{(n)}$, so that we obtain from $\delta v = 0$ an equation which may be written

$$V_n\phi^{(n)} + {}_nV\theta^{(n)} = 0,$$

which enables us to express $\theta^{(n)}$ in terms of $\phi^{(n)}$ and to write the important terms in $\int_{x'}^{x''} (\delta^2 u + \lambda\delta^2 v)\, dx$ in the form

$$\int_{x'}^{x''} \{{}_nV^2(U_{nn} + \lambda V_{nn}) - 2_nVV_n({}_nU_n + \lambda {}_nV_n) + V_n^2({}_{nn}U + \lambda_{nn}V)\} \frac{\phi^{(n)^2}}{{}_nV^2}\, dx.$$

Hence we see, by reasoning exactly similar to that in Prop. IV., that U cannot be a maximum unless the term in the bracket is of opposite sign to dx throughout the integration.

The proof in Props. V. and VI. applies without any modification.

Thus the problem reduces to the well-known one of finding the sign of a quadratic expression subject to linear relations among the variables.

9. When, however, the orders of the highest fluxions which appear in u and v are not identical, some care is required in ascertaining what the linear relation given by $\delta v = 0$ becomes. Thus, suppose that, omitting all but the highest fluxions, we take

$$\delta^2 u = A(\delta y^{(5)})^2 + B\delta\ddot{z}^2 + C\delta\ddot{w}^2,$$

$$\delta v = 0 = a\delta y^{(3)} + b\delta\dot{z} + c\delta w.$$

Here the highest fluxions in δv are lower than those of the corresponding variations in $\delta^2 u$, and we differentiate δv until one fluxion at least is common to the two. After one differentiation (neglecting all lower fluxions, by Prop. II.), we get, from $d\delta v = 0$,

$$ay^{(4)} + b\ddot{z} + c\dot{w} = 0,$$

so that $\ddot{z}$ cannot be large compared to both $y^{(4)}$ and $\dot{w}$, wherefore (Prop. II.) it must be negligible compared to $\delta y^{(5)}$ and $\delta\dot{w}$, which appear with it in $\delta^2 u$; in other words, the linear relation among the highest fluxions in $\delta^2 u$ in this case is $\delta\ddot{z} = 0$. That there is no linear relation between $y^{(5)}$ and $\dot{w}$ is easily seen; in fact, $\delta v = 0$ can only give rise to a single equation among the variations. Hence the conditions for a minimum are that Adx and Cdx shall both be positive.

If $\delta v = 0$ had been $a\delta\ddot{y} + b\delta z + c\delta w = 0$, then two differentiations would have given us $a\delta y^{(4)} + b\delta\ddot{z} + c\delta\dot{w} = 0$, so that, unless $\delta\ddot{z}$ and $\delta\dot{w}$ are large compared to $\delta y^{(4)}$ (in which case we write *in the limit* $b\delta\ddot{z} + c\delta\dot{w} = 0$), they must be negligible compared to $\delta y^{(5)}$ in $\delta^2 u$. Hence the linear relation now becomes $b\delta\ddot{z} + c\delta\dot{w} = 0$, and the conditions for a minimum are that Adx and $(c^2B + b^2C)\,dx$ shall be positive.

10. If $\delta v = 0$ had been $a\delta y^{(3)} + b\delta z^{(3)} + c\delta w^{(3)} = 0$, then $\delta\ddot{z}$ and $\delta\dot{w}$ cannot be comparable to $\delta y^{(5)}$ in $\delta^2 u$, unless $\delta y^{(3)}$ be negligible compared to $\delta z^{(3)}$ and $\delta w^{(3)}$ in $\delta v = 0$, *i.e.*, unless $b\delta z^{(3)} + c\delta w^{(3)} = 0$, in which case, integrating, $b\ddot{z} + c\dot{w} = 0$, *the constant vanishing, because both variations are zero at limits*, according to the supposition of the present paper. Hence we get the same conditions as before.

Thus, in every case, if we retain in $\delta v = 0$ only those fluxions which are highest *as compared with the fluxions of the same variations in* $\delta^2 u$, and then bring them to the same order by differentiation or integration (no constant being added), we obtain the linear relation which is to be used in connexion with $\delta^2 u$.*

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* The criteria given by M. Jordan in his *Cours d'Analyse* would, in all but the last example in the text, give no result at all, since the determinant equation for ρ ($I = 0$, Vol. III., p. 530) would vanish identically, and in the last example the only condition his method gives is A positive (since the other root of his determinant, $\rho = 0$, does not enter into the criteria). It is, however, obvious that A positive cannot be the only condition, for $\delta v = 0$ permits us to take $\delta y \equiv 0$ through a finite range of integration, provided $b\delta z + c\delta w \equiv 0$ also, and if B and C were both negative, it is obvious that U could not be a minimum. Similar difficulties arise with Clebsch's criteria.

The methods of Clebsch and Jordan depend wholly on analytical transformations,

When there are more variables or more equations of condition, a similar method is to be pursued. First ascertain what variations may be omitted from the final expression, and then ascertain what relations, if any, connect the magnitudes of those which we retain, and finally express the condition that $\delta^2 U$ shall not change sign. Evidently, too, the second condition, that which depends on the limits x' and x'' of the integration, is obtained by following the same rule as in the simpler case.

11. In some cases in which the limits are really fixed, it is nevertheless convenient to select as the independent variable a quantity whose limiting values are neither fixed nor arbitrary, but dependent as well on the form assumed for the solution as on the fixed limits of the other variables. Hence, when this form is varied, the limits of integration are also varied, and it is easy to see that $\delta^2 U$ does not then represent the whole of the part of ΔU which is quadratic in the variations. An example will best explain this.

EXAMPLE 1.—*Problem of Least Action.*

Let p be the type of the generalized coordinates, and $\dot{p}$ that of their velocities, T being the kinetic energy, and $T+V=0$ the form in which the equation of energy is written. Then

$$U=\int_{x'}^{x''}\left[T+\lambda(T+V)\right]dt$$

is the integral which has to be examined.

It is easily shown (see "Least Action" in Routh's *Rigid Dynamics*) that the part of ΔU which is linear in the variations is zero if the coor-

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the possibility of which is not demonstrated except by such loose methods as that of showing that the number of variables is the same as that of the equations to be satisfied. This method, which is never safe without further justification, is specially unsuited to the discussion of problems of maxima and minima, because we know *a priori* that it must lead to erroneous results after the "conjugate point" is reached, although that circumstance does not affect the *apparent* possibility of solving the equations. Probably the fault in M. Jordan's demonstration lies in this direction; thus, if

$$u\equiv A\dot{y}_1^2+B\dot{y}_2^2+C\dot{y}_3^2+D\dot{y}_4^2 \text{ and } v\equiv a\dot{y}_1+b\dot{y}_2+cy_3+dy_4,$$

it is impossible to solve his equations (4) of p. 506, because the five equations for five unknowns turn into three for two of the unknowns and two for the three remaining unknowns. Yet in this case the criteria he gives are correct, though the reasoning undoubtedly fails.

dinates satisfy the equations of Dynamics, *and if the value of* λ *be* $-\frac{1}{2}$. Hence, if the limits of t were fixed, Prop. IV. would apply immediately, because the highest variations, typified by $\delta\dot{p}$, appear as a positive function in $\delta^2 T$, and the quantity $(1+\lambda)\,dt$, which multiplies $\delta^2 T$, is essentially positive, being $\frac{1}{2}dt$.

But, since the limits of the independent variable are not fixed, this reasoning is not sufficient. In fact we have to find the sign of

$$\Delta U \equiv \int_{t'+\omega'}^{t''+\omega''} (u+\Delta u)\,dt - \int_{t'}^{t''} u\,dt,$$

where $$u = T+\lambda\,(T+V) = \tfrac{1}{2}\,(T-V),$$

since $$\lambda = -\tfrac{1}{2}.$$

Hence, expanding,

$$\Delta U = \int_{t'}^{t''} \Delta u\,dt + \Big|_{t'}^{t''} (u+\Delta u)\,\delta t$$

$$= \int_{t'}^{t''} (\delta u + \tfrac{1}{2}\delta^2 u)\,dt + \Big|_{t'}^{t''} u\,\delta t + \Big|_{t'}^{t''} \delta u\,\delta t \quad \ldots\ldots\ldots\ldots\ldots\ldots (5).$$

Taking first the part of this which is apparently linear in the variations first, and reducing in the ordinary way, we obtain

$$\delta U = \Sigma \Big|_{t'}^{t''} (1+\lambda)\,\frac{dT}{d\dot{p}}\,\delta p + \Big|_{t'}^{t''} T\delta t$$

$$+\Sigma \int_{t'}^{t''} \left\{ -(1+\lambda)\,\frac{d}{dt}\,\frac{dT}{d\dot{p}} - \frac{d\lambda}{dt}\,\frac{dT}{d\dot{p}} + (1+\lambda)\,\frac{dT}{dp} + \lambda\,\frac{dV}{dp} \right\} \delta p\,dt \quad \ldots(6).$$

Here $\delta p'$ is not the *complete* change in the limiting value of p; this complete change is

$$\Delta p' = \delta p' + \dot{p}'\delta t' + \tfrac{1}{2}\ddot{p}'\delta t'^2 + \&c.$$

If we substitute this value for $\delta p'$ in (6), with the similar value for $\delta p''$, we get (6) in a form containing terms linear in Δp and δt, together with a term

$$-\Big|_{t'}^{t''} (1+\lambda)\,\Sigma \left(\ddot{p}\,\frac{dT}{d\dot{p}} \right) \delta t^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots (7).$$

The linear terms in (6) vanish with stationary solution, so that the

only terms which remain over in (5) are the following, the terms in (7) being included,

$$\Delta U = \int_{t'}^{t''} \tfrac{1}{2}\delta^3 u\, dt + \Big|_{t'}^{t''} \delta u\, \delta t - \Big|_{t'}^{t''} (1+\lambda) \Sigma \left(\ddot{p}\frac{dT}{d\dot{p}}\right) \delta t^2 \ldots\ldots\ldots (8).$$

In finding the sign of (8), we cannot directly apply Prop. II., *since the limits of* δp, *&c. are not zero; but, if we take* $\delta t'$ *zero, we shall have* $\delta p'$ *zero*, since $\Delta p'$ then becomes equal to $\delta p'$. Hence Prop. II. applies, and therefore in (8) we may, *if the range of integration be short*, leave out the $\delta p'$s in $\delta^2 u$ in comparison with the $\delta\dot{p}$s, and $\frac{1}{2}\delta^3 u$ may be regarded as an essentially positive function of the $\delta\dot{p}$s, since it is derived from substituting $\delta\dot{p}$ for $\dot{p}$, &c. in T. Hence the first term in (8) is essentially positive. It remains to examine the terms depending on the limits.

Remembering that the range of integration is small, we may write

$$t''-t' < \frac{p''-p'}{\dot{p}_1},$$

where $\dot{p}_1$ is the *least* value of $\dot{p}$ (which however differs little from the greatest), and

$$\delta t'' < -\frac{p''-p'}{\dot{p}_1^2}\delta\dot{p}_2,$$

where $\delta\dot{p}_2$ is the greatest value of $\delta\dot{p}$. Therefore $\delta t''$ is of the order $(p''-p')\,\delta\dot{p}_2$, and $\delta t''^2$, which is of the order $(p''-p')^2\,\delta\dot{p}_2^2$, may be neglected in comparison with $\int_{t'}^{t''} \tfrac{1}{2}\delta^3 u\, dt$, which is of the order $\delta^2 u\,(t''-t')$ or $\delta^2 u\,(p''-p')$ or $\delta\dot{p}^2\,(p''-p')$, in which the $(p''-p')$ appears in the first order only. Hence we reject

$$-\Big|_{t'}^{t''} (1+\lambda) \Sigma \left(\ddot{p}\frac{dT}{d\dot{p}}\right) \delta t^2,$$

from (8).

Next as to $\Big|_{t'}^{t''} \delta u\, \delta t$. At first sight it would appear to be of the same order as $\int \tfrac{1}{2}\delta^3 u\, dt$, for $\Big|_{t'}^{t''} \delta u\, \delta t$ is of the order $\delta\dot{p}\,\delta t$ or $\delta\dot{p}\,(p''-p')\,\delta\dot{p}$, the same as that found for the integral. But the equation

$$\delta\,(T+V) = 0,$$

taken with Prop. II., shows that for a small range of integration δu

must vanish, or at least be of the same order as δV, which contains only δp, not $\delta \dot{p}$. Hence the term $\int_{t'}^{t''} \delta u \delta t$ may also be neglected in comparison with $\int_{t'}^{t''} \frac{1}{2}\delta^2 u dt$, which is therefore the only term to be retained in (8). Hence, for a short range of integration, the total variation of the integral is positive, and the stationary value is a true minimum. Props. V. and VI. at once enable us to prove that, until the kinetic focus is reached, the action is a true minimum, while, after it is reached, the action is neither a minimum nor a maximum.

Had it not been that the highest fluxions occurred in the same way both in the function to be integrated and in the equation of condition, we could not have reduced the problem to one in which we may consider the limits fixed, even when the range of integration is small. It will be shown in the paper on variable limits that in such cases we arrive analytically at the criterion in Prop. V. and § 5, relating to the "conjugate point."

12. There is a class of problems in which the quantity to be made a maximum or a minimum is taken as the independent variable s, and in this case the terms involving δs outside the limits in the complete second variation of U may be neglected, just as in the case in which the limits are fixed. For example, consider the following well-known problem, which will also serve to illustrate the application of the principles of § 2.

EXAMPLE 2.—*To draw between two points P and Q the shortest line of given curvature.**

Taking s as the independent variable, the integral to be made a minimum is $\int_Q^P ds$, and the limitation is that

$$\frac{1}{\rho^2} \text{ or } \left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2 + \left(\frac{d^2z}{ds^2}\right)^2$$

shall be constant, while, of course,

$$\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2 = 1.$$

* See Jellett, *Calculus of Variations*, pp. 194–202; Todhunter, *History of Calculus of Variations*, pp. 164–179; and Carll, *Calculus of Variations*, p. 405. None of these discussions go to the root of the matter, though the analytical work of obtaining the stationary solution is very fully given, especially by Mr. Todhunter.

Hence, using the fluxional notation, the integral we have to make a minimum is, according to the usual method,

$$\int_Q^P \left[1+\lambda(\dot{x}^2+\dot{y}^2+\dot{z}^2)+\mu(\ddot{x}^2+\ddot{y}^2+\ddot{z}^2)\right] ds,$$

the limits of x, y, and z being given, but those of $\dot{x}$, $\dot{y}$, and $\dot{z}$ being arbitrary.

It is easily found, and is shown in each of the works referred to, that the coefficients of the limiting values of $\delta\dot{x}$, $\delta\dot{y}$, and $\delta\dot{z}$ will not vanish unless the curve be plane; wherefore it would appear to follow that the solution must be the circular arc of the given radius which joins P and Q, since, if the curve be both plane and of constant curvature, it must be a circle.

If we apply the criterion of the foregoing pages, for *fixed* limits of x, y, z and $\dot{x}$, $\dot{y}$, $\dot{z}$, to this solution, we find that it gives a true minimum, for it is easy to see that μ does not change sign. In fact, if x', y', z' be taken as origin of coordinates,

$$\mu=\sqrt{(y''z-z''y)^2+(z''x-x''z)^2+(x''y-y''x)^2},$$

where each of the quantities in the bracket vanishes at each limit of integration, but not between. Furthermore, when we come to examine the terms in ΔU depending on $\delta s'$ and $\delta s''$, we find, exactly as in the case of least action, that they arise from

$$\int_{s'}^{s''}\delta u\, ds+\Big|_{s'}^{s''}(u\,\delta s+\delta u\,\delta s).$$

But in this case the stationary solution makes $\delta s''-\delta s'$ vanish to the first order of small quantities, so that if $\delta s'$ be taken as zero, $\delta s''$ vanishes to the first order, and $\delta s''^2$, as well as $\delta u''\delta s''$, may be neglected in considering the second variation. Hence the solution appears to be a true minimum for *fixed* limits of x, y, z, $\dot{x}$, $\dot{y}$, $\dot{z}$, and it is worth pointing out that the argument which shows that the terms of the second order depending on the variation of the independent variable may be neglected, is generally applicable to all problems in which that variable is made a maximum or a minimum.

But, if we endeavour to ascertain whether the circle is a true minimum when compared with other curves *having different tangents at P and Q*, that is, when the limits of $\dot{x}$, $\dot{y}$, and $\dot{z}$ are not fixed, we are unable to complete the solution on account of the impossibility of obtaining a general solution of the equations furnished by the

calculus. But, though unable to give a full account of the answer given by the calculus of variations to the question whether the circle is a minimum curve or not, we shall see presently, from geometrical considerations, what that answer must be.

Take any curve C in space, and generate a tubular surface by the motion of a circle of small radius r, whose centre is on the curve C, and whose plane is normal to the tangent to C. Then it is evident that we may trace on the surface so formed a line of any constant radius of curvature ρ, intermediate between r and the maximum radius of curvature of C, and that the curve so drawn will (or at least may) wind round the original curve. For instance, if C be a straight line, then the surface generated becomes a cylinder, and the line of constant radius of curvature ρ is a spiral, the interval h between whose threads is $\sqrt{\rho r - r^2}$. Now let us take r indefinitely small, still keeping ρ the same, and we find for h the expression $h^2 = \rho r$. Hence it follows at once that *a curve of any given curvature ρ may be drawn indefinitely near to a straight line, or to a line of any curvature of greater radius*, and it also follows from the expression for the length l of a single turn of the spiral, viz.,

$$l = \sqrt{h^2 + r^2} = h\sqrt{1 + \frac{r^2}{h^2}},$$

that, in the limit, $l = h$, and therefore *the length of the complete spiral is ultimately the same as that of the straight line with which it coincides;* and evidently the same conclusion follows if the axis of the tubular surface be itself curved. Hence the condition that the curvature is to be constant, though introduced in order to restrict the *length* of the curve, does not, in fact, put any restriction whatever on that *length*, and the answer to the question: "What is the shortest line of given curvature?" is: There is no definite shortest line of given curvature, no line differing finitely in position from the straight line joining the points, which line, regarded as a helix of the proper pitch, must be looked upon as a line of the given curvature, and therefore as the solution. Probably, however, no one would set the problem, if the irrelevancy of the restriction were apprehended.*

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* The following note is suggested by one of the referees:—

"The principle, of which the above example is a very apt illustration, that the existence of a least value of an integral does not carry with it the existence of a continuous path of integration which corresponds to that value, has been already emphasised by Weierstrass and others, in particular in connexion with questions of

The problem, however, deserves careful consideration, because it illustrates two important principles which are often forgotten. The first is the caution necessary in applying the argument (so often used to avoid the trouble of applying the criteria for discriminating maximum and minimum solutions) that, as we know there must be a minimum solution, and as the calculus of variations supplies only one stationary solution, that of the circle, we can conclude that the circle is the true minimum solution. We must be very certain that there is no other solution, definite, indefinite, or discontinuous, before we use this argument. In the present case, the result which led to the conclusion that the tangents at P and Q must lie in the same plane (whence it followed from the equation that the curve itself must be a plane curve throughout) might have been stated thus: So long as the tangents are not in the same plane, the value of the integral can be diminished,—a form of expression with which our solution, which in the *limit* is a straight line, entirely accords.

The second principle is brought to our notice when we discuss the sense in which the circle is a stationary solution. For here a difficulty meets us. Draw between P and Q a circle of radius ρ', slightly greater than the given value ρ; the curve so formed is therefore shorter, by a quantity of the first order, than the stationary solution obtained. Now wind a helix around the ρ' circle, so as to have the required radius of curvature ρ, and consider a variation given to the current values of x, y, and z, belonging to the stationary curve, such that the varied curve shall everywhere coincide with the curve obtained by the winding of the helix round the circle ρ'. The curve so obtained is indefinitely near the stationary curve; its radius of curva-

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determinacy in the theory of potentials and the theory of functions, where the demonstration has been based on the principle of Dirichlet.

"Thus the volume integral of Dirichlet,

$$\int dv \left\{ \left(\frac{d\phi}{dx}\right)^2 + \left(\frac{d\phi}{dy}\right)^2 + \left(\frac{d\phi}{dz}\right)^2 \right\},$$

certainly possesses a least value when it is subject to given values of ϕ at each point of the boundary; but that least value may conceivably be approached asymptotically by making the slope of the function ϕ increase more and more near a certain locus, so that the actual least value belongs to a function which is discontinuous along that locus.

"If the discontinuity entered in this way, the region of integration might be considered to be restricted so as to exclude it, and the difficulty might to that extent be got rid of. The example in the paper, however, gives a case in which the functional character of the solution breaks down over the whole range of the line of integration, and not merely in any particular part of it."

ture is everywhere of the correct value; and it is shorter, by a quantity of the *first* order (*i.e.*, the same order as the variations), than the supposed stationary curve, which would therefore seem not to be a stationary solution at all. Hence there is a difficulty to be explained.

The explanation is that the analytical statement of the problem, taken in connexion with the limitations stated in § 2, implies, not only that the radius of curvature is to be unchanged by the variation, but also that the quantities $\frac{d^2x}{ds^2}$, $\frac{d^2y}{ds^2}$, $\frac{d^2z}{ds^2}$ are to receive *small changes only*, a limitation not hinted at in the verbal statement of the problem, and one which prevents our using the analytical method to compare the stationary curve with all possible consecutive curves of the same curvature. For instance, we cannot compare it with a curve whose projection on the plane of xy has points of inflexion where that of the stationary curve has none; for in that case the variations of $\dot{x}$, $\dot{y}$, $\ddot{x}$, $\ddot{y}$ (or some of them) must be finite, and the distinction in order of magnitude between the first and the second variation disappears. It does not appear possible to give an analytical statement of the problem (except perhaps by quaternions) which shall be free from this difficulty, though in all probability, if we could complete the analytical solution by finding the value of the integral for varied limits of $\dot{x}$, $\dot{y}$, $\dot{z}$, we should then find analytically that the circle did not give a really stationary value to the integral.

The solution here given must not be confounded with that suggested, on other grounds, by Mr. Todhunter in his *History*, which was that a plane curve, of the required radius of curvature, could be obtained by joining together arcs of circles of the required radius, the said arcs to touch each other and to have their convexities turned opposite ways. If the proposer of the problem is willing to accept such a solution as this, with its discontinuous changes of curvature from positive to negative values (evidenced by the existence of what, to the eye, appear as points of inflexion), well and good; but if not, the solution is disposed of as soon as the proposer of the problem observes that there must be a number of curves of *really constant* curvature between the points, and that it is from among such curves (of the given radius of curvature) that he wants the shortest to be chosen.

Part IV.

13. *Discrimination of Maxima and Minima Values of Functions of one or more Integrals, as $F(U, V, W)$, where U, V, and W are Integrals of known form.*

This is an interesting class of problem which seems to have been hardly discussed at all, even in regard to the stationary solution. An example will serve to explain the theory :—

Example 3.—*A uniform chain, length undetermined, is hung from two fixed points (P and Q) ; determine its form and length so that the ratio of the length of the chain to the distance of its centre of mass from a given line equidistant from the points shall be a maximum or a minimum.*

Taking the given line as axis of x, and writing $+a$, y_0, and $-a$, y_0, as the coordinates of P and Q, we have to make

$$\frac{\int_{-a}^{a}\sqrt{1+\dot{y}^2}\,dx}{\frac{\int_{-a}^{a}y\sqrt{1+\dot{y}^2}\,dx}{\int_{-a}^{a}\sqrt{1+\dot{y}^2}\,dx}} \quad \text{or} \quad \frac{\left(\int_{-a}^{a}\sqrt{1+\dot{y}^2}\,dx\right)^2}{\int_{-a}^{a}y\sqrt{1+\dot{y}^2}\,dx}$$

a maximum or a minimum. For shortness, denote $\int_{-a}^{a}\sqrt{1+\dot{y}^2}\,dx$ by L, and $\int_{-a}^{a}y\sqrt{1+\dot{y}^2}\,dx$ by M; then, evidently,

$$\Delta\frac{\left(\int_{-a}^{a}\sqrt{1+\dot{y}^2}\,dx\right)^2}{\int_{-a}^{a}y\sqrt{1+\dot{y}^2}\,dx} \equiv \Delta\frac{L^2}{M} \equiv \delta\frac{L^2}{M}+\tfrac{1}{2}\delta^2\frac{L^2}{M}+\&c.,$$

where

$$\delta\frac{L^2}{M} = \frac{L}{M^2}(2M\delta L - L\delta M),$$

and

$$\delta^2\frac{L^2}{M} = \frac{L}{M^2}(2M\delta^2 L - L\delta^2 M) + 2\frac{M^2(\delta L)^2 - 2LM\delta L.\delta M + L^2(\delta M)^2}{M^3} \quad \ldots(9).$$

Hence, to secure the vanishing of the first variation, we have

$$2M\delta\int_Q^P\sqrt{1+\dot{y}^2}\,dx - L\delta\int_Q^P y\sqrt{1+\dot{y}^2}\,dx = 0 \quad \ldots\ldots\ldots\ldots(10),$$

in which L and M, being the values of the integrals for the stationary solution, are constants whose values are to be ascertained. The solution of (10) is obtained by integrating the equation

$$-L\sqrt{1+\dot{y}^2}-\frac{d}{dx}\left\{(2M-Ly)\frac{\dot{y}}{\sqrt{1+\dot{y}^2}}\right\}=0,$$

which gives
$$y=\frac{2M}{L}+\frac{c}{2}(e^{x/c}+e^{-x/c}) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (11),$$

in which c is a constant of integration, the other constant being zero owing to the symmetry of the curve. We can now easily calculate L and M.

$$L=\int_{-a}^{a}\sqrt{1+\dot{y}^2}\,dx=\int_{-a}^{a}\tfrac{1}{2}(e^{x/c}+e^{-x/c})\,dx=c\,(e^{a/c}-e^{-a/c}) \ldots (12),$$

while for M we obtain the equation

$$M=\int_{-a}^{a}y\sqrt{1+\dot{y}^2}\,dx=\tfrac{1}{2}\int_{-a}^{a}\left\{\frac{2M}{L}+\frac{c}{2}(e^{x/c}+e^{-x/c})\right\}(e^{x/c}+e^{-x/c})\,dx$$

$$=2M+\frac{c^2}{4}(e^{2a/c}-e^{-2a/c})+ca,$$

from which, with (12), we obtain

$$\frac{M}{L}=-\frac{4a+c\,(e^{2a/c}-e^{-2a/c})}{4\,(e^{a/c}-e^{-a/c})}.$$

Substituting this in the expression for y in (11), and putting $x=a$ and $y=y_0$, we obtain an equation for c,

$$y_0=-\frac{2a}{e^{a/c}-e^{-a/c}}.$$

It is evident that there is but one real value of c which satisfies this equation, and that, if y_0 be positive, the value of c is negative, being, in fact,

$$\frac{a}{\log_e\left(\dfrac{\sqrt{y_0^2+a^2}-a}{y_0}\right)}.$$

Turning now to (9), we observe that the second variation consists of two classes of terms, typified by $\delta^2 A$ and $\delta A\,\delta L$ or δL^2, which are both of the second order in the variations, and which, if the range of the integration were finite, would be of the same order of magnitude.

But, by following the method of Prop. IV., and considering only a short range of integration $(x''-x')$, it is evident that, whereas $\delta^2 A$ is of the same order as $(x''-x')(\delta\dot{y})^2$, $\delta A\,\delta L$ is of the order $(x''-x')^2(\delta\dot{y})^2$, and may therefore be neglected in comparison with $\delta^2 A$ or $\delta^2 L$, a quite unexpected advantage of taking a short range of integration. Hence, with this restriction, the important part of the second variation is got from the terms involving $\delta\dot{y}^2$ in

$$\frac{L}{M^2}(2M\delta^2 L - L\delta^2 M),$$

and is

$$\frac{L^2}{M^2}\int_{x'}^{x''}\left(\frac{2M}{L}-y\right)\frac{\delta\dot{y}^2}{(1+\dot{y}^2)^{\frac{3}{2}}}\,dx.$$

Now, from (11),

$$\frac{2M}{L}-y=-\frac{c}{2}(e^{x/c}+e^{-x/c}),$$

and is always positive, since c is negative. Hence, so far as this part of the criterion is concerned, the value of $\frac{L^2}{M}$ is a minimum for the stationary solution.

As to the second part of the criterion, it is evident that Props. V. and VI. apply to this class of problem, and, since it has been shown that there is but one value for c, it is evident that no second consecutive curve of the system can be drawn, anywhere between $x=a$ and $x=-a$, so as to intersect the stationary curve; and therefore the solution is a true minimum.*

14. It is not difficult to see that if, in any problem involving func-

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* In this case there is a solution which does not seem to be obtainable from the equations supplied by the calculus of variations, at least when we take y and x as dependent and independent variables. If y_0 be positive, it appears evident that, by taking c an indefinitely small positive quantity, we get infinite values for L and M, while for $\frac{M}{L}$ we get $\left(-\frac{L}{4}+y_0\right)$ *plus* some small positive quantity which diminishes indefinitely as L increases. (This is evident from observing that, if L be infinite, the catenary becomes practically two parallel lines, and that $\frac{M}{L}$ is the distance of the centre of mass above the axis of x.) Hence $\frac{M}{L^2}$ has $-\frac{1}{4}$ for a minimum value when the chain hangs in a catenary, and it approaches this value when the length is infinite; and, evidently, if the form be altered from the catenary to another curve, the length being unchanged, the centre of mass is brought nearer the axis. Hence $\frac{M}{L^2}=-\frac{1}{4}$ is a true minimum value of the fraction for all possible forms of the chain, and therefore -4 is a maximum value for the reciprocal which we are discussing. The value c = negative zero does not give a maximum or minimum unless $y_0=0$.

It is only in the case of $y_0=0$ that the equations furnished by the calculus of

tions of integrals, we impose the restriction that another integral is to be constant, or, more generally, is to be a given function of the limits, we do not introduce any additional theoretic difficulty, although the actual solution of the problem may transcend our power. Nor would there be any further theoretic difficulty if there were more dependent variables, connected by equations of condition, except so far as this might prevent our treating the problem as one of "fixed limits." In many cases, such as Example 1 (least action) and Example 2 (line of shortest curvature), the difficulty which thus arises may be evaded, and, where that is not possible, we find (as mentioned at the end of § 9) that the criterion obtained includes that relative to the "conjugate point" in Prop. V. and § 5.

On an Operator that produces all the Covariants and Invariants of any System of Quantics.

By W. E. Story, of Worcester, Mass., U.S.A. Received 19th April, 1892. Read May 12th, 1892.

I assume a system of quantics of orders m, m', m'', &c. respectively, in the n variables $x, y, z, \ldots$, to be given thus:

$$\Sigma \frac{m!}{g!\,h!\,i!\ldots} a_{g,h,i,\ldots} x^g y^h z^i \ldots \quad [g+h+i+\ldots = m],$$

$$\Sigma \frac{m'!}{g!\,h!\,i!\ldots} b_{g,h,i,\ldots} x^g y^h z^i \ldots \quad [g+h+i+\ldots = m'],$$

$$\Sigma \frac{m''!}{g!\,h!\,i!\ldots} c_{g,h,i,\ldots} x^g y^h z^i \ldots \quad [g+h+i+\ldots = m''],$$

&c.,

and call the literal coefficients $a_{g,h,i,\ldots}$, $b_{g,h,i,\ldots}$, $c_{g,h,i,\ldots}$, &c., freed from the numerical polynomial coefficients, simply the *coefficients* of the

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variations give both solutions satisfactorily, and this appears to be on account of the fact that we have no means of distinguishing between the length of the curve from P to Q, and the length between the points where it cuts the axis of x, for the abscissæ of the latter points, in terms of which the length is obtained, are also $+a$ and $-a$ if the length be infinite. Hence, whatever be the real value of y_0, the work of the problem for an infinite chain gives the same result as if y_0 were zero, and thus we always seem to obtain two infinite solutions, one a maximum and the other a minimum, while, in fact, the minimum is the only true infinite solution.

several quantics. To any coefficient I give a *weight relatively to each variable;* namely, the weight of any coefficient relatively to either variable is the sum of all its suffices excepting that suffix that corresponds to the variable in question; thus, the weight of $c_{g,h,i,\ldots}$ relatively to y is $g+i+\ldots = m''-h$. I suppose each variable to have the weight 1 relatively to itself, and the weight 0 relatively to any other variable. The weight of any product of coefficients and variables relatively to any variable is, then, the aggregate weight of all the factors of the product relatively to the variable in question. If the terms of any polynomial are all of the same weight relatively to *each* of the variables involved in the given quantics (but not necessarily of one and the same weight relatively to the *several* variables), the polynomial is *isobaric.*

We have here to do only with holomorphic functions that are homogeneous in the coefficients of each of the given quantics and homogeneous in the variables; such functions I call simply *homogeneous functions* of coefficients and variables.

For convenience, I include covariants and invariants under the one name "covariants," defining a covariant to be a homogeneous function of the coefficients and variables that is simply multiplied by a power of the determinant of substitution when any linear transformation is imposed on the variables.

I consider two classes of simple transformations, which I call expansions and shears.

An *expansion* relatively to either variable is a linear transformation by which that variable receives a constant multiplier, the *modulus* of the expansion, while all other variables remain unchanged.

A *shear* relatively to any pair of variables (taken in a certain order) is a linear transformation by which the second of these variables receives as increment a constant multiple of the first, while all other variables remain unchanged.

Thus the substitution of ρx for x is an expansion relatively to x, and the substitution of $x+\lambda y$ for x is a shear relatively to yx.

I. It is easy to prove that *every linear transformation can be compounded of expansions and shears.*

A covariant is, then, a homogeneous function that is not altered by any shear, and is simply multiplied by a power of the *modulus* of expansion in consequence of any expansion.

It appears at once that a covariant of *degrees* $j, j', j'', \ldots$ in the coefficients of the several quantics and of *order* ϑ in the n variables is multiplied, in consequence of any linear transformation, by a power

of the determinant of substitution whose exponent is $\frac{1}{n}(\Sigma mj - \vartheta)$; while any isobaric part of it, of weights $w_x, w_y, w_z, \ldots$ relatively to the several variables, is multiplied, in consequence of any expansion relatively to x, by a power of the modulus of expansion whose exponent is $\Sigma mj - w_x$; so that $w_x = \frac{1}{n}\left[(n-1)\Sigma mj + \vartheta\right]$, and $w_y, w_z, \ldots$ have the same value as w_x; therefore *a covariant is necessarily isobaric of one and the same weight relatively to all the several variables.*

II.

A homogeneous function of coefficients and variables that is not altered by any shear relatively to yx shall be called a *differentiant relatively to* yx. The condition that ϕ shall be a differentiant relatively to yx is that

$$\widehat{yx} \,.\, \varphi \equiv 0, \tag{1}$$

shall be satisfied identically, where the differential operator $\widehat{yx}$ is defined by

$$\left.\begin{aligned} \widehat{yx} \equiv \quad & \Sigma h \,.\, a_{g+1,h-1,i,\ldots} \frac{\partial}{\partial a_{g,h,i,\ldots}} + \Sigma h \,.\, b_{g+1,h-1,i,\ldots} \frac{\partial}{\partial b_{g,h,i,\ldots}} \\ & + \Sigma h \,.\, c_{g+1,h-1,i,\ldots} \frac{\partial}{\partial c_{g,h,i,\ldots}} + \&c. - y\frac{\partial}{\partial x}, \end{aligned}\right\} \tag{2}$$

each summation extending to all admissible values of $g, h, i, \ldots$, and the several sums $+\&c.$ referring to the several quantics of the system.*

A homogeneous function that is a differentiant relatively to every pair of variables of a given set may be called a *complete differentiant* relatively to the variables of that set. It appears, then, that *every isobaric differentiant relatively to all the variables involved in the given system of quantics is a covariant of that system; and, conversely, every covariant is an isobaric differentiant relatively to all the variables.*

III.

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* The name "differentiant" was employed by Professor Sylvester in the case of binary quantics (*Phil. Mag.*, 1878); and the operator $\widehat{yx}$ was defined by Professor Cayley ("Mémoire sur les hyper-déterminants," *Crelle's Jour.*, Vol. xxx.), and represented by the symbols $\Sigma\{y\partial_x\} - y\partial_x$, X, and $\square$ for binary quantics in subsequent papers.

The *shear-symbols* $\widehat{yx}$, &c., when applied to any homogeneous function ϕ of weights w_x, w_y, w_z, ... relatively to the several variables are combined or interchanged by these formulæ

$$(3)\quad \begin{cases} \widehat{yx}\,.\,\widehat{xy}^{a}\,.\,\phi \equiv \widehat{xy}^{a}\,.\,\widehat{yx}\,.\,\phi + a\,(w_y - w_x - a + 1)\,.\,\widehat{xy}^{a-1}\,.\,\phi, \\ \widehat{yx}\,.\,\widehat{xz}^{a}\,.\,\phi \equiv \widehat{xz}^{a}\,.\,\widehat{yx}\,.\,\phi - a\,.\,\widehat{xz}^{a-1}\,.\,\widehat{yz}\,.\,\phi, \\ \widehat{yx}\,.\,\widehat{zy}^{a}\,.\,\phi \equiv \widehat{zy}^{a}\,.\,\widehat{yx}\,.\,\phi + a\,.\,\widehat{zy}^{a-1}\,.\,\widehat{zx}\,.\,\phi, \\ \widehat{yx}\,.\,\widehat{zx}\,.\,\phi \equiv \widehat{zx}\,.\,\widehat{yx}\,.\,\phi, \qquad \widehat{yx}\,.\,\widehat{yz}\,.\,\phi \equiv \widehat{yz}\,.\,\widehat{yx}\,.\,\phi, \\ \widehat{yx}\,.\,\widehat{zs}\,.\,\phi \equiv \widehat{zs}\,.\,\widehat{yx}\,.\,\phi, \end{cases}$$

where x, y, z, s are any four different variables.

IV. It is readily proved that *a homogeneous isobaric function* ϕ, *that is of one and the same weight relatively to the several variables* $x, y, z, \ldots, u, v$ *of any set, is a complete differentiant relatively to the variables of this set, if only*

$$\widehat{xy}\,.\,\phi \equiv 0, \quad \widehat{xz}\,.\,\phi \equiv 0, \quad \ldots, \quad \widehat{xu}\,.\,\phi \equiv 0, \quad \widehat{xv}\,.\,\phi \equiv 0,$$

or

$$\widehat{yx}\,.\,\phi \equiv 0, \quad \widehat{zx}\,.\,\phi \equiv 0, \quad \ldots, \quad \widehat{ux}\,.\,\phi \equiv 0, \quad \widehat{vx}\,.\,\phi \equiv 0.*$$

The operator $\widehat{yx}$ evidently increases by 1 the weight relatively to y, diminishes by 1 the weight relatively to x, and leaves unaltered the weights relatively to the other variables, as well as the degrees and order, of any homogeneous isobaric function to which it is applied.

Let $x, y, z, \ldots, u, v$ be any set of variables selected at pleasure from the variables involved in the given quantics, and let v be any one of the variables of this set. Let ϕ be any homogeneous isobaric function of the coefficients of the given quantics, and of any or all of the variables involved in them, of weights w_x, w_y, w_z, ..., w_u, w_v relatively to $x, y, z, \ldots, u, v$ respectively.

The operator

$$\widehat{xv}^{\alpha}\,.\,\widehat{yv}^{\beta}\,.\,\widehat{zv}^{\gamma}\,\ldots\,\widehat{uv}^{\epsilon}\,.\,\widehat{vu}^{\epsilon}\,\ldots\,\widehat{vz}^{\gamma}\,.\,\widehat{vy}^{\beta}\,.\,\widehat{vx}^{\alpha}$$

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* See Salmon's *Modern Higher Algebra*, 4th ed., p. 136. I suppose this theorem is due to Professor Cayley, but I cannot say where he has proved it for more than two variables.

alters neither the weights relatively to the several variables, the degrees, nor the order of the function ϕ, when applied to it. This operator, it may be remarked, has two "alæ,"

$$\widehat{xv}^{\alpha}.\widehat{yv}^{\beta}.\widehat{zv}^{\gamma}\dots\widehat{uv}^{\epsilon}, \text{ and } \widehat{vu}^{\epsilon}\dots\widehat{vz}^{\gamma}.\widehat{vy}^{\beta}.\widehat{vx}^{\alpha},$$

in either of which any two shear-symbols have only the letter v in common, and that occurs in the first place in both factors or in the second place in both; by the fourth and fifth formulæ of (3), any two factors of either "ala" are, then, commutative; *i.e.*, the order of the variables $x, y, z, \dots, u$ in the operator is immaterial. Multiplying this operator by a numerical factor $C_{\alpha,\beta,\gamma,\dots,\epsilon}$, to be determined later, and taking the sum of the product for all integral but not negative values of $\alpha, \beta, \gamma, \dots, \epsilon$, we find an operator

$$(4)\quad Q \equiv \Sigma C_{\alpha,\beta,\gamma,\dots,\epsilon}.\widehat{xv}^{\alpha}.\widehat{yv}^{\beta}.\widehat{zv}^{\gamma}\dots\widehat{uv}^{\epsilon}.\widehat{vu}^{\epsilon}\dots\widehat{vz}^{\gamma}.\widehat{vy}^{\beta}.\widehat{vx}^{\alpha},$$

in which we may neglect all terms excepting those for which $0 \leqq \alpha \leqq w_x$, $0 \leqq \beta \leqq w_y$, $0 \leqq \gamma \leqq w_z$, ..., $0 \leqq \epsilon \leqq w_u$, when it is applied to ϕ.

By formulæ (3), we have

$$(5)\quad \left\{\begin{aligned} &\widehat{yx}.\widehat{xv}^{\alpha}.\widehat{yv}^{\beta}.\widehat{zv}^{\gamma}\dots\widehat{uv}^{\epsilon}.\widehat{vu}^{\epsilon}\dots\widehat{vz}^{\gamma}.\widehat{vy}^{\beta}.\widehat{vx}^{\alpha}.\phi \\ \equiv\ &\beta.\widehat{xv}^{\alpha}.\widehat{yv}^{\beta}.\widehat{zv}^{\gamma}\dots\widehat{uv}^{\epsilon}.\widehat{vu}^{\epsilon}\dots\widehat{vz}^{\gamma}.\widehat{vy}^{\beta-1}.\widehat{vx}^{\alpha+1}.\phi \\ &-\alpha.\widehat{xv}^{\alpha-1}.\widehat{yv}^{\beta+1}.\widehat{zv}^{\gamma}\dots\widehat{uv}^{\epsilon}.\widehat{vu}^{\epsilon}\dots\widehat{vz}^{\gamma}.\widehat{vy}^{\beta}.\widehat{vx}^{\alpha}.\phi \\ &+\widehat{xv}^{\alpha}.\widehat{yv}^{\beta}.\widehat{zv}^{\gamma}\dots\widehat{uv}^{\epsilon}.\widehat{vu}^{\epsilon}\dots\widehat{vz}^{\gamma}.\widehat{vy}^{\beta}.\widehat{vx}^{\alpha}.\widehat{yx}.\phi, \end{aligned}\right.$$

and analogous expressions for the results of applying $\widehat{zx}, \dots, \widehat{ux}$ to the general term of $Q.\phi$; and

$$(6)\quad \left\{\begin{aligned} &\widehat{vx}.\widehat{xv}^{\alpha}.\widehat{yv}^{\beta}.\widehat{zv}^{\gamma}\dots\widehat{ux}^{\epsilon}.\widehat{vu}^{\epsilon}\dots\widehat{vz}^{\gamma}.\widehat{vy}^{\beta}.\widehat{vx}^{\alpha}.\phi \\ \equiv\ &\alpha(w_v-w_x+\alpha+1).\widehat{xv}^{\alpha-1}.\widehat{yv}^{\beta}.\widehat{zv}^{\gamma}\dots\widehat{uv}^{\epsilon}.\widehat{vu}^{\epsilon}\dots\widehat{vz}^{\gamma}.\widehat{vy}^{\beta}.\widehat{vx}^{\alpha}.\phi \\ &+\beta^2.\widehat{xv}^{\alpha}.\widehat{yv}^{\beta-1}.\widehat{zv}^{\gamma}\dots\widehat{uv}^{\epsilon}.\widehat{vu}^{\epsilon}\dots\widehat{vz}^{\gamma}.\widehat{vy}^{\beta-1}.\widehat{vx}^{\alpha+1}.\phi+\&\text{c.} \\ &+\beta.\widehat{xv}^{\alpha}.\widehat{yv}^{\beta-1}.\widehat{zv}^{\gamma}\dots\widehat{uv}^{\epsilon}.\widehat{vu}^{\epsilon}\dots\widehat{vz}^{\gamma}.\widehat{vy}^{\beta}.\widehat{vx}^{\alpha}.\widehat{yx}.\phi+\&\text{c.} \\ &+\widehat{xv}^{\alpha}.\widehat{yv}^{\beta}.\widehat{zv}^{\gamma}\dots\widehat{uv}^{\epsilon}.\widehat{vu}^{\epsilon}\dots\widehat{vz}^{\gamma}.\widehat{vy}^{\beta}.\widehat{vx}^{\alpha+1}.\phi, \end{aligned}\right.$$

where +&c. after any term represents an aggregate of terms formed from that term by interchanging y with z, ..., u, and at the same time interchanging β with γ, ..., ϵ, successively.

If, then, $\quad \widehat{yx}\,.\,\phi \equiv 0, \quad \widehat{zx}\,.\,\phi \equiv 0, \quad \ldots, \quad \widehat{ux}\,.\,\phi \equiv 0,$

we have

$$\widehat{xy}\,.\,Q\,.\,\phi \equiv \Sigma A^{(yx)}_{\alpha,\beta,\gamma,\ldots,\epsilon}\,.\,\widehat{xv}^{\alpha}\,.\,\widehat{yv}^{\beta+1}\,.\,\widehat{zv}^{\gamma}\ldots\widehat{uv}^{\epsilon}\,.\,\widehat{vu}^{\epsilon}\ldots\widehat{vz}^{\gamma}\,.\,\widehat{vy}^{\beta}\,.\,\widehat{vx}^{\alpha+1}\,.\,\phi,$$

where $\quad A^{(yx)}_{\alpha,\beta,\gamma,\ldots,\epsilon} = (\beta+1)\,.\,C_{\alpha,\beta+1,\gamma,\ldots,\epsilon} - (\alpha+1)\,.\,C_{\alpha+1,\beta,\gamma,\ldots,\epsilon},$

and similar expressions for $\widehat{zx}\,.\,Q\,.\,\phi$, ..., $\widehat{ux}\,.\,Q\,.\,\phi$; and

$$\widehat{vx}\,.\,Q\,.\,\phi \equiv \Sigma A^{(vx)}_{\alpha,\beta,\gamma,\ldots,\epsilon}\,.\,\widehat{xv}^{\alpha}\,.\,\widehat{yv}^{\beta}\,.\,\widehat{zv}^{\gamma}\ldots\widehat{uv}^{\epsilon}\,.\,\widehat{vu}^{\epsilon}\ldots\widehat{vz}^{\gamma}\,.\,\widehat{vy}^{\beta}\,.\,\widehat{vx}^{\alpha+1}\,.\,\phi,$$

where

$$\begin{aligned} A^{(vx)}_{\alpha,\beta,\gamma,\ldots,\epsilon} = {} & (\alpha+1)(w_v - w_x + \alpha + 2)\,C_{\alpha+1,\beta,\gamma,\ldots,\epsilon} + (\beta+1)^2\,.\,C_{\alpha,\beta+1,\gamma,\ldots,\epsilon} \\ & + (\gamma+1)^2\,.\,C_{\alpha,\beta,\gamma+1,\ldots,\epsilon} + \ldots + (\epsilon+1)^2\,.\,C_{\alpha,\beta,\gamma,\ldots,\epsilon+1} \\ & + C_{\alpha,\beta,\gamma,\ldots,\epsilon}\,. \end{aligned}$$

We shall, then, have

$$\widehat{yx}\,.\,Q\,.\,\phi \equiv 0, \quad \widehat{zx}\,.\,Q\,.\,\phi \equiv 0, \quad \ldots, \quad \widehat{ux}\,.\,Q\,.\,\phi \equiv 0, \quad \widehat{vx}\,.\,Q\,.\,\phi \equiv 0,$$

if $\quad (\alpha+1)\,.\,C_{\alpha+1,\beta,\gamma,\ldots,\epsilon} = (\beta+1)\,.\,C_{\alpha,\beta+1,\gamma,\ldots,\epsilon} = (\gamma+1)\,.\,C_{\alpha,\beta,\gamma+1,\ldots,\epsilon}$

$$= \ldots = (\epsilon+1)\,.\,C_{\alpha,\beta,\gamma,\ldots,\epsilon+1},$$

and

$$(\alpha+1)\left\{w_v - w_x + (\alpha+2) + (\beta+1) + (\gamma+1) + \ldots + (\epsilon+1)\right\}\,.\,C_{\alpha+1,\beta,\gamma,\ldots,\epsilon}$$

$$= -\,C_{\alpha,\beta,\gamma,\ldots,\epsilon};$$

i.e., if

(7)

$$C_{\alpha,\beta,\gamma,\ldots,\epsilon} = (-1)^{\alpha+\beta+\gamma+\ldots+\epsilon}\,\frac{(w_r - w_x + \nu)!\;C_{0,0,0,\ldots,0}}{\alpha!\,\beta!\,\gamma!\ldots\epsilon!\,(\alpha+\beta+\gamma+\ldots+\epsilon+w_v-w_x+\nu)!},$$

where ν is the number of variables x, y, z, ..., u (without v).

Putting $w_x = w_y = w_z = \ldots = w_u = w_v$ and (for convenience) $C_{0,0,0,\ldots,0} = 1$, we obtain an operator

(8)

$$\left\{ \begin{array}{l} \{x, y, z, \ldots, u\,;\, v\} \\ \equiv \Sigma\,(-1)^\sigma \dfrac{\nu!}{\alpha!\,\beta!\,\gamma!\ldots\epsilon!(\sigma+\nu)!}\,.\,\widehat{xv}^\alpha\,.\,\widehat{yv}^\beta\,.\,\widehat{zv}^\gamma \ldots \widehat{uv}^\epsilon\,.\,\widehat{vu}^\epsilon \ldots \widehat{vz}^\gamma\,.\,\widehat{vy}^\beta\,.\,\widehat{vx}^\alpha, \end{array} \right.$$

(where σ is written for $\alpha+\beta+\gamma+\ldots+\epsilon$ in each term), which turns any homogeneous isobaric function ϕ, of one and the same weight relatively to all the several variables $x, y, z, \ldots, u, v$ for which

$$\widehat{yx}\,.\,\phi \equiv 0, \quad \widehat{zx}\,.\,\phi \equiv 0, \quad \ldots, \quad \widehat{ux}\,.\,\phi \equiv 0,$$

into a homogeneous isobaric function ψ, of the same weights, degrees, and order, for which

$$\widehat{yx}\,.\,\psi \equiv 0, \quad \widehat{zx}\,.\,\psi \equiv 0, \quad \ldots, \quad \widehat{ux}\,.\,\psi \equiv 0, \quad \widehat{vx}\,.\,\psi \equiv 0\,;$$

V. *i.e.*, by Theorem IV., *the operator* $\{x, y, z, \ldots, u\,;\, v\}$, *defined by* (8), *turns any isobaric complete differentiant relatively to the variables* $x, y, z, \ldots, u$, *of the same weight relatively to* v *as relatively to* $x, y, z, \ldots, u$, *into an isobaric complete differentiant relatively to* $x, y, z, \ldots, u, v$, *of the same weights, degrees, and order.*

Now it is evident from (6) (on putting $\beta = \gamma = \ldots = \epsilon = 0$, and writing y instead of v) or directly from (3) that

$$\widehat{yx}\,.\,\widehat{xy}^\alpha\,.\,\widehat{yx}^\alpha\,.\,\phi \equiv \alpha\,(w_y - w_x + \alpha + 1)\,.\,\widehat{xy}^{\alpha-1}\,.\,\widehat{yx}^\alpha\,.\,\phi + \widehat{xy}^\alpha\,.\,\widehat{yx}^{\alpha+1}\,.\,\phi,$$

and therefore, if ϕ is of one and the same weight relatively to x and y,

$$\widehat{yx}\,.\,\widehat{xy}^\alpha\,.\,\widehat{yx}^\alpha\,.\,\phi \equiv \alpha\,(\alpha+1)\,.\,\widehat{xy}^{\alpha-1}\,.\,\widehat{yx}^\alpha\,.\,\phi + \widehat{xy}^\alpha\,.\,\widehat{yx}^{\alpha+1}\,.\,\phi,$$

and

$$\widehat{yx}\,.\,\{x\,;\,y\}\,.\,\phi \equiv \widehat{yx}\,.\,\Sigma\,(-1)^\alpha \frac{\widehat{xy}^\alpha\,.\,\widehat{yx}^\alpha}{\alpha!\,(\alpha+1)!}\,.\,\phi \equiv 0,$$

as Hilbert has proved.*

Moreover, the operator $\{x, y, z, \ldots, u\,;\, v\}$ leaves any complete differentiant relatively to the variables $x, y, z, \ldots, u, v$ unaltered, since the result of operating upon any such complete differentiant

* See *Math. Annalen*, Vol. XXXVI., p. 523.

with any term of $\{x, y, z, \ldots, u; v\}$ for which $\alpha, \beta, \gamma, \ldots, \epsilon$ are not all 0 vanishes identically, and the term of this operator for which $\alpha = \beta = \gamma = \ldots = \epsilon = 0$ is 1.

Therefore, if we write

$$(9)\quad [\phi]_1 \equiv \{x, y, z, \ldots, u; v\}\{x, y, z, \ldots; u\}\ldots\{x, y; z\}\{x; y\}.\phi,$$

we have the theorem ···

VI. $[\phi]_1$ *is a covariant of the given system of quantics and of the same type as* ϕ (*i.e.*, of the same weights, degrees, and order), *provided only that* ϕ *is homogeneous and isobaric of one and the same weight relatively to all the several variables, is the most general covariant of this type if* ϕ *is the most general homogeneous isobaric function of the type, and is* ϕ *itself if* ϕ *is a covariant of the system.*

If the given quantics are homogeneous in each of several sets of variables subject to independent linear transformations, the operator $[\ ]_1$ formed with the variables of any such set is commutative with that formed with the variables of any other such set, and the covariants of the system are found by applying the product of the operators $[\ ]$, formed with the variables of all the several sets to functions that are homogeneous in the coefficients of each quantic and in each set of variables, isobaric relatively to each variable of whatever set, and of one and the same weight relatively to the several variables of each set.

There is another operator that answers the same purpose as $[\ ]_1$ (perhaps identical with it in its effect upon the functions to which we apply it) that I designate by $[\ ]'$; namely, if [using the notation of (8) and (9)]

(10)

$$\left\{\begin{array}{l}\{v; u, \ldots, z, y, x\} \\ \equiv \Sigma(-1)^{\sigma}\dfrac{\nu!}{\alpha!\beta!\gamma!\ldots\epsilon!(\sigma+\nu)!}.\widehat{vx}^{\alpha}.\widehat{vy}^{\beta}.\widehat{vz}^{\gamma}\ldots\widehat{vu}^{\epsilon}.\widehat{uv}^{\epsilon}\ldots\widehat{zv}^{\gamma}.\widehat{yv}^{\beta}.\widehat{xv}^{\alpha},\end{array}\right.$$

$$(11)\quad [\phi]' \equiv \{v; u, \ldots, z, y, x\}\{u; \ldots, z, y, x\}\ldots\{z; y, x\}\{y; x\}.\phi.$$

Thursday, June 8th, 1892.

Prof. GREENHILL, F.R.S., President, in the Chair.

Messrs. G. T. Bennett, J. E. Campbell, and J. L. Hatton were admitted into the Society.

Letters were read from Messrs. Poincaré, Hertz, Mittag Leffler, Schwarz, and Beltrami, acknowledging receipt of letter informing them of their election as foreign members, and thanking the Society for the compliment paid them.

Prof. Henrici exhibited a model of movable hyperboloids of one sheet. In 1873, he gave a student at University College the problem to construct a model of a hyperboloid of one sheet, by fixing three sticks anyhow, placing others so as to cut these, and tying them together wherever they met. He told the student that the system would soon become rigid, but was surprised that this was not the case. It was easy to see the reason of this fact, and thus he established the theorem:—If the two sets of generators of a hyperboloid be connected by articulated joints wherever they meet, then the system remains movable, the hyperboloid changing its shape. It was also soon found that each point moves during this deformation along the normal to the momentary position of the surface, and that therefore the different positions of the surface constitute a system of confocal hyperboloids. He then made a model such that the generators represented by sticks meet at points which lie on lines of curvature of the hyperboloid. These, therefore, describe confocal ellipsoids and hyperboloids of two sheets. In January, 1874, he exhibited this model at a meeting of the Mathematical Society. Soon afterwards a student made two copies of this model, and these were fastened together in such a manner that both could move together, remaining always confocal. It was this model which was now shown. The properties of the movable hyperboloid became more widely known through a question which Prof. Greenhill set in 1878 at the Mathematical Tripos Examination, and this led Prof. Cayley to give a solution of it in the *Messenger of Mathematics.* Since that time many French mathematicians have made further investigations of the property in question; Darboux and Mannheim in particular have

made beautiful applications of the deformable hyperboloid to the motion of a gyratory rigid body.

The following further communications were made:—

The Second Discriminant of the Ternary Quantic $x'u+y'v+z'w$: Mr. J. E. Campbell.

On the Reflection and Refraction of Light from a Magnetized Transparent Medium: Mr. A. B. Basset.

Note on Approximate Evolution: Prof. Lloyd Tanner.

A Proof of the Exactness of Cayley's Number of Seminvariants of a Given Type: Mr. E. B. Elliott.

Further Note on Automorphic Functions: Prof. W. Burnside.

The following presents were received:—

"Beiblätter zu den Annalen der Physik und Chemie," Band XVI., Stück 5.

"Journal of the Institute of Actuaries," No. CLXV., April, 1892.

"Memoirs and Proceedings of the Manchester Literary and Philosophical Society," Vol. v., No. 1, 1891–2.

"Nyt Tidsskrift for Mathematik," A. Tredje Aargang, No. 3.

"Nyt Tidsskrift for Mathematik," B. Tredje Aargang, No. 2; Copenhagen, 1892.

"Bulletin of the New York Mathematical Society," Vol. I., No. 8; May, 1892.

"Bulletin des Sciences Mathématiques," Tome XVI., Avril, 1892; Paris.

"Atti della Reale Accademia dei Lincei—Rendiconti," Vol. I., Fasc. 6, 7, 8, 1 Sem.; Rome, 1892.

"Educational Times," June, 1892.

"Journal für die reine und angewandte Mathematik," Band CIX., Heft 4; Berlin, 1892.

"A Treatise on the Mathematical Theory of Elasticity," A. E. H. Love, Vol. I., 8vo; Cambridge, 1892. From the Author.

"On the Effects of Selection—An Actuarial Essay," Emory McClintock, 8vo; New York, 1892. From the Author.

"Reprint of a Description of a Machine for Finding the Numerical Roots of Equations and tracing a variety of useful Curves," F. Bashforth, 8vo pamphlet; Cambridge, 1892. From the Author.

Note on the Algebraic Theory of Elliptic Transformation. By JOHN GRIFFITHS, M.A. Received and read 10th December, 1891. Received in revised form June 28th, 1892.

In a former note, published in the *Proceedings of the London Mathematical Society* (Vol. XVIII., pp. 377–88), it was proved by the present writer that the transformations with regard to the Θ functions can be easily deduced from an elementary formula, viz., if f is a function of x and k, then

$$\frac{d^2f}{dx\,dk} = \frac{d^2f}{dk\,dx}.$$

I now propose to show that we can form the differential equation satisfied by the numerator and denominator of a transformation equation

$$y = \frac{P(x, k)}{Q(x, k)}$$

by means of the elementary theorem in question.

SECTION I. *Elementary Theorem.*

If
$$\frac{M\,dy}{\sqrt{1-y^2 . 1-\lambda^2y^2}} = \frac{dx}{\sqrt{1-x^2 . 1-k^2x^2}},$$

$$M^{-2} = \frac{nkk'^2}{\lambda\lambda'^2}\frac{d\lambda}{dk} \ (n \text{ being a number}),$$

$$\Omega \equiv nkk'^2\frac{d}{dk} + \Delta x\, G(x, k)\frac{d}{dx}, \quad \Delta x = \sqrt{1-x^2 . 1-k^2x^2},$$

$$\Omega\left(\frac{1-y^2}{1-\lambda^2y^2}\right) = 0,$$

i.e.,
$$\Omega y = M^{-2}\lambda^2 y\,(1-y^2);$$

then
$$\Delta x\frac{dG}{dx} + \frac{nk^3k'^2x^2}{1-k^2x^2} - \frac{nkk'^2}{M}\frac{dM}{dk} = M^{-2}\lambda^2\,(1-y^2) \ldots\ldots\ldots\ldots (1).$$

This result is practically the same as that given in my former note (Vol. XVIII., p. 383), and depends upon the identity

$$\frac{d^2y}{dx\,dk} = \frac{d^2y}{dk\,dx}.$$

By taking $$G(x, k) = \Delta x \frac{d}{dx} \log \frac{Q}{(1-k^2x^2)^{\frac{1}{2}n}},$$

we arrive immediately, by means of this proposition, at the Θ transformation equation

$$\Theta\{M^{-1}(n+u_0), \lambda\} = Ce^{\mu u^2+\nu u}\Theta^n(u)\, Q(\mathrm{sn}\, u),$$

where C, μ, ν are k-functions.

Assuming then that

$$G(x, k) = \Delta x \frac{d}{dx} \log \frac{Q}{(1-k^2x^2)^{\frac{1}{2}n}},$$

the equation (1) may be written in the form

$$nk^2(1-x^2) - x(1+k^2-2k^2x^2)\frac{Q'}{Q} + (\Delta x)^2\left\{\frac{Q''}{Q} - \left(\frac{Q'}{Q}\right)^2\right\} - \frac{nkk'^2}{M}\frac{dM}{dk}$$
$$= M^{-2}\lambda^2(1-y^2) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

where $$Q' = \frac{dQ}{dx}, \quad Q'' = \frac{d^2Q}{dx^2}.$$

Section 2. *The Method of Squares.*

If P and Q be rational and integral functions of x and k, the equation (2) of the previous section leads to what may be called the method of squares.

In fact we have

$$\left(M^{-2}\lambda^2 + n\frac{kk'^2}{M}\frac{dM}{dk} - nk^2\right)Q^2 + nk^2x^2Q^2 + x(1+k^2-2k^2x^2)\,Q'Q$$
$$+ (\Delta x)^2(Q'^2 - Q''Q) = M^{-2}\lambda^2P^2.$$

In a similar manner, by writing $\frac{1}{\lambda y}$ for y, and taking

$$G(x, k) = \Delta x \frac{d}{dx} \log \frac{P}{(1-k^2x^2)^{\frac{1}{2}n}},$$

in the formulæ of Section I., it follows that

$$\left(M^{-2}\lambda^2 + \frac{nkk'^2}{M}\frac{dM}{dk} - nk^2\right)P^2 + nk^2x^2P^2 + x(1+k^2-2k^2x^2)\,P'P$$
$$+ (\Delta x)^2(P'^2 - P''P) = M^{-2}Q^2.$$

For example, if $n = 3$,

$$P = b_1 x + b_3 x^3, \quad Q = 1 + a_2 x^2,$$

we find that $$a_2^4 - 6k^2 a_2^2 - 4k^3(1+k^3)\, a_2 - 3k^4 = 0.$$

SECTION 3. *Differential Equation satisfied by P and Q.*

Taking $$G(x, k) = \Delta x \frac{d}{dx} \log \frac{Q}{(1-k^2x^2)^{\frac{1}{2}n}},$$

and writing $$\xi = M^{-2}\lambda^2(1-y^2),$$

the results of Section I. are

$$\xi = M^{-2}\lambda^2(1-y^2)$$

$$= nk^2(1-x^2) + x(2k^2x^2 - 1 - k^2)\frac{Q'}{Q} + 1 - x^2 \,.\, 1 - k^2x^2 \frac{d}{dx}\left(\frac{Q'}{Q}\right) - \frac{nkk'^2}{M}\frac{dM}{dk},$$

$$\Omega\left(\frac{1-y^2}{1-\lambda^2 y^2}\right) = \Omega\left\{\frac{\xi}{\lambda^2\xi + M^{-2}\lambda'^2}\right\} = 0,$$

where $$\lambda'^2 = 1 - \lambda^2,$$

$$\left.\begin{aligned} \Omega\xi &= 2\xi\left\{\xi - \frac{nkk'^2}{M}\frac{dM}{dk} + M^{-2}(1-2\lambda^2)\right\} \\ \Omega &\equiv nkk'^2\frac{d}{dk} + \left\{nk^2x(1-x^2) + 1 - x^2 \,.\, 1 - k^2x^2\frac{Q'}{Q}\right\}\frac{d}{dx} \end{aligned}\right\} \quad \ldots(3).$$

We thus find ultimately that Q satisfies a differential equation of the form

$$X_1 Q^3 + X_2 Q^2 + X_3 Q$$

$$+ x_3\left\{x_3\frac{d^2Q}{dx^2} + (x_2 + 2xx_1)\frac{dQ}{dk} + 2nkk'^2\frac{dQ}{dk}\right\}\left(\frac{dQ}{dx}\right)^2 = 0 \quad \ldots(4),$$

where $x_1 = nk^2(1-x^2)$, $x_2 = x(2k^2x^2 - 1 - k^2)$, $x_3 = 1 - x^2 \,.\, 1 - k^2x^2$, and X_1 is a rational and integral function of degree 4 in x, also X_2, X_3 are rational and integral functions of x involving the differential coefficients $\frac{dQ}{dx}$, $\frac{dQ}{dk}$, $\frac{d^2Q}{dx^2}$, &c.

If, then, Q is a rational and integral function of x of the form

$$Q = a_0 + a_1 x + a_2 x^2 + \ldots,$$

which contains no square factor, it follows from (4) that the expression

$$x_3\frac{d^2Q}{dx^2}+(x_2+2xx_1)\frac{dQ}{dx}+2nkk'^2\frac{dQ}{dk} \text{ involves the factor } Q.$$

In other words, we may take

$$x_3\frac{d^2Q}{dx^2}+(x_2+2xx_1)\frac{dQ}{dx}+2nkk'^2\frac{dQ}{dk}=(\alpha+\beta x+\gamma x^2)\,Q\ldots\ldots(5),$$

where $x_3=1-x^2.\,1-k^2x^2$, $x_2=x\,(2k^2x^2-1-k^2)$, $x_1=nk^2\,(1-x^2)$, and α, β, γ are k-functions. In general we have

$$\beta=0 \quad \text{and} \quad \gamma=-n\,(n-1)\,k^2.$$

SECTION 4. *Example of the result of Section* 3.

If
$$P=b_1x+b_3x^3+\ldots+b_nx^n,$$

and
$$Q=1+a_2x^2+a_4x^4+\ldots+a_tx^t+\ldots a_{n-1}x^{n-1},$$

where n is an odd number, the differential equation (5) of the last section gives the following result, viz.,

$$(t+2)(t+1)\,a_{t+2}+2nkk'^2\frac{da_t}{dk}-\left\{2a_2+t^2-t\,(2n-t)\,k^2\right\}a_t$$
$$+(n-t+2)(n-t+1)\,k^2a_{t-2}=0,$$

where
$$2a_2=M^{-2}\lambda^2-nk^2+\frac{nkk'^2}{M}\,\frac{dM}{dk}.$$

It thus appears that, if a_2 were given, we could find a_4, a_6, ... a_{n-1}.

For example, if n is a square number and $\lambda=k$, then $a_2=0$ and a_4, a_6, ... are also known functions of k.

It is clear that in general we have a differential equation of a certain order to determine a_2.

For instance, if $n=3$, so that $a_4=0$, $a_0=0$, ..., we have, by putting $t=2$, $a_0=1$,

$$3kk'^2\frac{da_2}{dk}-a_2^2-2\,(1-2k^2)\,a_2+3k^2=0.$$

Similarly, if $n=5$, so that $a_6=0$, ... there results a differential equation of the second order in k to determine a_2.

The above differential equation connecting a_{t+2}, a_t, a_{t-2} is also true when n is any number, and Q is an integral function of the form

$$Q = 1 + a_1 x + a_2 x^2 + \ldots + a_n x^n,$$

the general term being $a_t x^t$.

If Q be written

$$Q = 1 + \sqrt{k}\, a_1 x + k a_2 x^2 + \ldots + k^{\frac{1}{2}t} a_t x^t + \ldots + k^{\frac{1}{2}n} a_n x^n,$$

and

$$k + \frac{1}{k} = p,$$

we have

$$(t+4)(t+3)\, a_{t+4} + 2nk^{\frac{1}{2}} \frac{d}{dk}(a_{t+2}) - \left\{2a_2 - (t+2)(n-t-2)\, p\right\} a_{t+2} + (n-t)(n-t-1)\, a_t = 0.$$

For instance, if $Q = 1 + \sqrt{k}\, a_1 x + k a_2 x^2$,

we have $n = 2$, $a_3 = 0$, $a_4 = 0$, ...; so that, when $t = -1$, $a_{-1} = 0$, we obtain

$$4k^{\frac{1}{2}} \frac{da_1}{dk} = (2a_2 - p)\, a_1.$$

Again, when $t = 0$, $\quad 4k^{\frac{1}{2}} \frac{da_2}{dk} - 2a_2^2 + 2 = 0.$

It thus appears that $a_2 = 1$, $a_1 = ck^{-\frac{1}{2}}(1+k)^{\frac{1}{2}}$ (where c is a constant independent of k) are solutions.

Section 5. *Deduction of the P Formulæ.**

The following is an example of a method by which the P formulæ may be derived from those for Q.

Let $\quad Q = 1 + a_1 x + a_2 x^2 + \ldots + a_t x^t + \ldots + a_n x^n,$

$$P = b_0 + b_1 x + b_2 x^2 + \ldots + b_t x^t + \ldots + b_n x^n,$$

and suppose the transformation equation $y = \frac{P}{Q}$ to be such that y is

* This section has been added at the suggestion of one of the referees.

changed into $\frac{1}{\lambda y}$ when $\frac{1}{kx}$ is written for x; we then have

$$(1)\quad (t+2)(t+1)\,a_{t+2}+2nkk'^2\frac{da_t}{dk}-\{2a_2+t^2-t\,(2n-t)\,k^2\}\,a_t$$
$$+(n-t+2)(n-t+1)\,k^2a_{t-2}=0,$$

$$(2)\quad (t+2)(t+1)\,b_{t+2}+2nkk'^2\frac{db_t}{dk}-\{2a_2-M^{-2}\lambda'^2+t^2-t\,(2n-t)\,k^2\}\,b_t$$
$$+(n-t+2)(n-t+1)\,k^2b_{t-2}=0.$$

In the particular example under consideration it can be proved without any difficulty that either of these formulæ may be transformed into the other.

For, if $$y=\frac{b_0+b_1x+\ldots+b_nx^n}{1+a_1x+\ldots+a_nx^n},$$

and also $$\frac{1}{\lambda y}=\frac{b_0+b_1\,(kx)^{-1}+\ldots+b_n\,(kx)^{-n}}{1+a_1\,(kx)^{-1}+\ldots+a_n\,(kx)^{-n}},$$

or $$y=\frac{k^nx^n+a_1k^{n-1}x^{n-1}+\ldots+a_n}{\lambda\{b_0k^nx^n+b_1k^{n-1}x^{n-1}+\ldots+b_n\}},$$

we must have

$$\left.\begin{array}{l|l} b_n=\mu k^n & a_n=\mu\lambda k^nb_0 \\ b_{n-1}=\mu k^{n-1}a_1 & a_{n-1}=\mu\lambda k^{n-1}b_1 \\ \ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots \\ b_t=\mu k^ta_{n-t} & a_t=\mu\lambda k^tb_{n-t} \end{array}\right\}\quad\text{and}\quad \mu^2\lambda k^n=1.$$

Hence, from (2), it follows that

$$(t+2)(t+1)\,\mu k^{t+2}a_{n-t-2}+2nkk'^2\frac{d}{dk}(\mu k^ta_{n-t})$$
$$-\{2a_2-M^{-2}\lambda'^2+t^2-t\,(2n-t)\,k^2\}\,\mu k^ta_{n-t}$$
$$+(n-t+2)(n-t+1)\,k^2\cdot\mu k^{t-2}a_{n-t+2}=0,$$

or $$(n-t+2)(n-t+1)\,a_{n-t+2}+2nkk'^2\frac{da_{n-t}}{dk}$$
$$-\left\{2a_2-M^{-2}\lambda'^2+t^2-t\,(2n-t)\,k^2-\frac{2nkk'^2}{\mu k^t}\,\frac{d}{dk}(\mu k^t)\right\}a_{n-t}$$
$$+(t+2)(t+1)\,k^2a_{n-t-2}=0;$$

where $$\mu^2\lambda k^n = 1,$$

and, consequently, $$\mu^2 k^{2t} = \frac{1}{\lambda k^{n-2t}};$$

$$-nkk'^2 \frac{d}{dk}\log(\mu^2 k^{2t}) = nkk'^2 \frac{d}{dk}\log(\lambda k^{n-2t}) = \frac{nkk'^2}{\lambda}\frac{d\lambda}{dk} + n(n-2t)k'^2$$
$$= M^{-2}\lambda'^2 + n(n-2t)k'^2.$$

It thus appears that (2) is transformed into

$$(n-t+2)(n-t+1)a_{n-t+2} + 2nkk'^2\frac{da_{n-t}}{dk}$$
$$-\{2a_2+(n-t)^2-(n^2-t^2)k^2\}a_{n-t} + (t+2)(t+1)k^2 a_{n-t-2} = 0.$$

Now, if we write t for $n-t$, this is

$$(t+2)(t+1)a_{t+2} + 2nkk'^2\frac{da_t}{dk} - \{2a_2+t^2-t(2n-t)k^2\}a_t$$
$$+(n-t+2)(n-t+1)k^2 a_{t-2} = 0;$$

in other words (2) can be transformed into (1).

Further Note on Automorphic Functions. By W. Burnside.

Received and read June 9th, 1892.

I propose in the present paper to continue the consideration of certain groups, and the automorphic functions connected with them, with which I dealt shortly in a previous paper, published in the current volume of the Society's *Proceedings*. In that paper it was shown that, for symmetrical fuchsian groups of the first class, automorphic functions taking every value twice only in the generating (or any) polygon can always be found, and therefore that the algebraic equation connecting two different automorphic functions is in this case an equation of the hyperelliptic class.

I shall here show how, when the group is given, to calculate the coefficients in the equation, incidentally expressing the two variables which the equation connects as uniform functions of a single parameter.

Herr H. Weber, in a paper published in the *Göttingen Nachrichten*, 1886, with the title "Ein Beitrag zu Poincaré's Theorie der Fuchs'schen Functionen," has already dealt with the problem of expressing the variables in a hyperelliptic equation as uniform functions of a parameter, and the mode of expression obtained in this paper is not essentially different from M. H. Weber's; still, though in this point possessing no novelty, the forms of the expressions here obtained are both infinite series and infinite products, while M. Weber's functions are all expressed in the form of infinite products, and moreover the group on which the functions depend is simpler than (in fact is a sub-group of) the group from which M. Weber starts.

The latter group is defined as follows:—

Take n circles with their centres lying on the real axis, and such that each one is outside all the others. Then n elliptic substitutions of period 2, with the n pairs of points where the circles meet the axis as their double points, are the generating substitutions of M. Weber's group. Any substitution of the group can therefore be represented by an even number of inversions at the system of $n+1$ circles formed by the n given circles and the axis of x. The symmetrical fuchsian group whose substitutions are the equivalents of an even number of inversions at the n circles is immediately seen to be a self-conjugate sub-group of the former, and it is on this group that the functions involved really depend.

After obtaining the equation connecting two independent automorphic functions, I go on to determine explicitly the relations between the integrals of the first species to which the equation gives rise, and the ϕ-functions of the corresponding group.

In the second paragraph, applying directly the definition which Prof. Klein gives in dealing with Abelian functions of a new element which he has introduced into the theory, I have shown how to express any automorphic function as the product of a number of factors which bear the same relation to it that a linear factor bears to a rational function.

Lastly, the relations between a symmetrical fuchsian group and its simplest sub-group (which is also symmetrical), as well as those between the functions connected with the two groups, are investigated; and, in the particular case when the hyperelliptic integrals reduce to elliptic by a quadratic substitution, the form of the generating polygon is determined.

1. *On the Algebraic Equation connecting Two Independent Automorphic Functions.*

I suppose for the present that the symmetrical fuchsian group is defined by $n+1$ circles $C_0, C_1, C_2 \ldots C_n$, each outside all the others and with their centres on the axis of x. If C_p be taken also to represent an inversion with respect to the p^{th} circle, the n fundamental substitutions of the group may be written

$$S_p = C_p C_0 \qquad (p = 1, 2, \ldots n).$$

If α, β are the double points of S_p, and α_i, β_i the homologues of α, β with respect to any substitution of the group, it was shown in my former paper that

$$\theta(z, J_p) = \sum_i \left(\frac{1}{z-\alpha_i} - \frac{1}{z-\beta_i}\right),$$

the summation extending to all *different* pairs of points α_i, β_i.

If α_i, β_i are the homologues of α, β for the substitution S_i, and α_j, β_j, for the substitution $C_0 S_i C_0$, which is clearly also a substitution of the group, then, since C_0 changes α into β, and S_i changes β into β_i, it follows that α_j and β_i are inverse points with respect to C_0, as also are β_j and α_i; hence, pairing the substitutions in this way,

$$\theta(z, J_p) = \frac{1}{z-\alpha} - \frac{1}{z-\beta} + \Sigma' \left[\frac{1}{z-\alpha_i} - \frac{1}{z-\beta_j} + \frac{1}{z-\alpha_j} - \frac{1}{z-\beta_i}\right].$$

Now, if P, Q are inverse points on a diameter MN of a circle, it may be easily verified that

$$\int_M^N \left(\frac{1}{z-P} - \frac{1}{z-Q}\right) dz = \pm \pi i,$$

where the integration is taken along the semicircle from M to N.

Hence, if a_0, b_0 be the points where C_0 meets the real axis, the above form of $\theta(z, J_p)$, taken in conjunction with the previous result that the integral is always a finite quantity, gives

$$\int_{a_0}^{b_0} \theta(z, J_p)\, dz = (2m+1)\pi i,$$

where m is an integer, the integration being conducted along the

circle C_0; and therefore

$$\exp\left[\phi_p(b_0)-\phi_p(a_0)\right]=-1,$$

where S_p is any one of the fundamental substitutions.

If, instead of taking for the generating polygon the region external to $C_1, C_2, \ldots C_n$, and to their inverses in C_0, there be taken the region external to $C_0, C_1, \ldots C_{i-1}, C_{i+1} \ldots C_n$ and to their inverses in C_i, the group remains the same, while C_i takes the place of C_0 in the previous investigation, and the substitutions

$$C_pC_i \quad (p=0, 1, \ldots i-1, i+1, \ldots n)$$

must be regarded as the fundamental substitutions. Expressed in terms of the former fundamental substitutions, these are

$$S_i^{-1}, S_1S_i^{-1} \ldots S_{i-1}S_i^{-1}, S_{i+1}S_i^{-1} \ldots S_nS_i^{-1}.$$

If, then, a_i, b_i are the points where C_i meets the real axis, and J is the homologue of infinity for any one of the n substitutions just written,

$$\int_{a_i}^{b_i}\theta(z, J)\,dz=(2m+1)\pi i,$$

m again being an integer. Now it has been proved before that

$$\theta(z, J_{p^{-1}q})=\theta(z, J_q)+\theta(z, J_{p^{-1}}).$$

Hence, if $q \neq i$,

$$\theta(z, J_q)=\theta(z, J_{i^{-1}q})-\theta(z, J_{i^{-1}}),$$

and

$$\int_{a_i}^{b_i}\theta(z, J_q)\,dz=2m'\pi i;$$

while

$$\int_{a_i}^{b_i}\theta(z, J_{i^{-1}})\,dz=-\int_{a_i}^{b_i}\theta(z, J_i)\,dz=(2m''+1)\pi i.$$

Hence, finally,

$$\exp\left[\phi_i(b_i)-\phi_i(a_i)\right]=-1,$$

$$\exp\left[\phi_q(b_i)-\phi_q(a_i)\right]=1, \qquad q \neq i.$$

It will be convenient to use the abbreviation z_i for $\frac{\alpha_i z+\beta_i}{\gamma_i z+\delta_i}$. With

this notation it was shown in the previous paper that

$$\chi_{a,b}(z_p)-\chi_{a,b}(z)=\phi_p(b)-\phi_p(a),$$

and now, if for a, b in this formula a_0, b_0, a_1, b_1, &c. be successively written, there results

$$\exp\chi_{a_0,b_0}(z_p)=-\exp\chi_{a_0,b_0}(z),\quad p=1,\ 2\ldots n,$$

and

$$\exp\chi_{a_i,b_i}(z_p)=\exp\chi_{a_i,b_i}(z),\quad p=1,2\ldots i-1,\ i+1\ldots n,$$

$$\exp\chi_{a_i,b_i}(z_i)=-\exp\chi_{a_i,b_i}(z).$$

Hence $\exp 2\chi_{a_i,b_i}(z)$ is an automorphic function, and, since in the generating polygon it has a single double infinity at b_i and a single double zero at a_i, its square root $\exp\chi_{a_i,b_i}(z)$ is the analogue of a hyperelliptic function.

Also the above formulæ show that

$$\prod_0^n \exp\chi_{a_i,b_i}(z)$$

is unaltered by the fundamental substitutions, and hence this also is an automorphic function.

I return now to functions in the form of infinite series, and, to simplify the analysis as far as possible, I suppose the circle C_0 to be the axis of y, which, as I have shown, involves no real loss of generality.

Then, as shown on pp. 76, 77 of my former paper,

$$x_a=\psi_a(z)-\psi_{-a}(z)+\text{constant}$$

is an automorphic function which takes every value twice in the generating polygon. It will first be proved directly that between any two such functions a lineo-linear equation holds.

Replacing $\psi_a(z)$ and $\psi_{-a}(z)$ by the forms given on p. 68, and omitting a constant term, which if necessary can be replaced in the result,

$$x_a=\theta(a,z)-\theta(-a,z)$$

$$=\theta(a,z)+\theta(a,-z).$$

Hence

$$x_a x_b=\sum_i\sum_j(\gamma_i a+\delta_i)^{-2}(\gamma^j b+\delta_j)^{-2}\left[\frac{1}{a_i-z}+\frac{1}{a_i+z}\right]\left[\frac{1}{b_i-z}+\frac{1}{b_i+z}\right].$$

On resolving the product of the two brackets into partial fractions,

the term depending on $\frac{1}{a_i - z}$ is

$$\sum_i \sum_j (\gamma_i a + \delta_i)^{-2} (\gamma_j b + \delta_j)^{-2} \frac{1}{a_i - z} \left[\frac{1}{b_j - a_i} + \frac{1}{b_j + a_i} \right]$$

$$= \sum_i \frac{(\gamma_i a + \delta_i)^{-2}}{a_i - z} \left[\theta(b, a_i) + \theta(b, -a_i) \right]$$

$$= \sum_i \frac{(\gamma_i a + \delta_i)^{-2}}{a_i - z} \left[\theta(b, a) + \theta(b, -a) \right]$$

$$= \theta(a, z) \left[\theta(b, a) + \theta(b, -a) \right].$$

Hence

$$x_a x_b = x_a \left[\theta(b, a) + \theta(b, -a) \right] + x_b \left[\theta(a, b) + \theta(a, -b) \right],$$

or

$$x_b = \frac{x_a \left[\theta(b, a) + \theta(b, -a) \right]}{x_a - \left[\theta(a, b) + \theta(a, -b) \right]}.$$

Now

$$x_a = 2\sum \frac{(\gamma_i a + \delta_i)^{-2} a_i}{a_i^2 - z^2},$$

so that x_a has a double zero at $z = \infty$, and, taking account of this, the above relation between x_a and x_b is what would have been obtained by assuming the one a linear function of the other from other considerations, and determining the constants from the zero and infinities.

The two infinities of x_a are in general distinct, since, if a is in the generating polygon, $-a$ is generally a different point also in the generating polygon. But, of the two circles C_p and C'_p, only one can be reckoned as belonging to the generating polygon, and hence, if a coincides with one of the points a_i or b_i, the two separate infinities of x_a coalesce into one double infinity. From this it follows that x_{b_i} / x_{a_i} can only differ from $\exp 2\chi_{a_i, b_i}(z)$ by a constant factor. Since C_0 is here taken for the axis of y, b_0 and a_0 are 0 and ∞.

Now, when a diminishes without limit,

$$\text{Lt. } x_a = \text{Lt.} \sum_i \left(\frac{(\gamma_i a + \delta_i)^{-2}}{\frac{\alpha_i a + \beta_i}{\gamma_i a + \delta_i} - z} - \frac{(\gamma_i a - \delta_i)^{-2}}{\frac{\alpha_i a - \beta_i}{\gamma_i a - \delta_i} - z} \right)$$

$$= \text{Lt.} \sum_i \left(\frac{1}{a - \frac{\delta_i z - \beta_i}{-\gamma_i z + \alpha_i}} - \frac{1}{a + \frac{\delta_i}{\gamma_i}} + \frac{1}{a + \frac{\delta_i z - \beta_i}{-\gamma_i z + \alpha_i}} - \frac{1}{a - \frac{\delta_i}{\gamma_i}} \right)$$

$$= 2a \sum_i \left[\left(\frac{\gamma_i}{\alpha_i} \right)^2 - \left(\frac{\gamma_i z + \delta_i}{\alpha_i z + \beta_i} \right)^2 \right];$$

and therefore $$x = \sum_i \left[\left(\frac{\gamma_i}{a_i} \right)^2 - \left(\frac{\gamma_i z + \delta_i}{a_i z + \beta_i} \right)^2 \right]$$

is an automorphic function taking every value twice, with a double infinity at the origin and a double zero at infinity; and if the value of the argument is expressed by writing $x(z)$, it follows that

$$\exp 2\chi_{\infty, 0}(z) = Cx,$$

and $$\exp 2\chi_{a_i, b_i}(z) = C' \frac{x - x(a_i)}{x - x(b_i)},$$

where C, C' are constants.

Finally, then, the two linearly-independent automorphic functions x and y, where

$$y = \exp \prod_0^n \chi_{a_i, b_i}(z),$$

are connected by the equation

$$y^2 = x \prod_1^n \frac{x - x(a_i)}{x - x(b_i)},$$

where the constants are directly calculable by using particular values of z in the series

$$\sum_i \left[\left(\frac{\gamma_i}{a_i} \right)^2 - \left(\frac{\gamma_i z + \delta_i}{a_i z + \beta_i} \right)^2 \right].$$

It is not without interest now to verify directly that the n independent hyperelliptic integrals of the first kind connected with the last equation are identical with n linearly independent ϕ-functions of the group.

The integrals may be written

$$\int \frac{dx}{[x - x(b_i)]\, y} \quad (i = 1.2 \ldots n),$$

and it is therefore to be shown that

$$\frac{1}{[x - x(b_i)]\, y} \frac{dx}{dz}$$

can be expressed in the form

$$\sum_{i=1}^{i=n} c_i \theta(z, J_i).$$

For this purpose it is necessary to determine the zeros and infinities of $\frac{dx}{dz}$.

Now, if a, b are any one of the pairs of points a_i, b_i, then

$$\frac{x-x(a)}{x-x(b)} = \exp 2\chi_{a,b}(z);$$

and therefore

$$\left[x(a)-x(b)\right]\frac{dx}{dz}$$
$$= \left[x-x(a)\right]\left[x-x(b)\right]\sum_i(\gamma_i z+\delta_i)^{-2}\left\{\frac{1}{z_i-a}-\frac{1}{z_i-b}\right\}.$$

When $z=a$, $x-x(a)$, considered as a function of z, has a double zero; and therefore $\frac{dx}{dz}$ has single zeros at the points a, b.

Also, since $$\frac{dx}{dz} = \sum_i \frac{(\gamma_i z+\delta_i)^{-2}}{z_i^3},$$

it has a triple infinity at $z=0$, and this is its only independent infinity; and, since x itself has a double zero at infinity, $\frac{dx}{dz}$ must have a triple zero there. This enumeration is complete, for it was shown, pp. 78, 79 of my former paper, that the number of zeros of any function of the form,

$$\sum_i f(z_i)(\gamma_i z+\delta_i)^{-2}$$

in the generating polygon exceeded that of the infinities by $2n$, and while $\frac{dx}{dz}$ has a single triple infinity it has also been shown to have a triple zero and $2n$ simple zeros at the points a_1, b_1 ... a_n, b_n.

Considered as a function of z,

$$\frac{1}{\left[x-x(b_i)\right]y}$$

or $$\frac{\left[x-x(b_1)\right]\ldots\left[x-x(b_{i-1})\right]\left[x-x(b_{i+2})\right]\ldots\left[x-x(b_n)\right]}{\sqrt{x\left[x-x(a_1)\right]\left[x-x(b_1)\right]\ldots\left[x-x(a_n)\right]\left[x-x(b_n)\right]}}$$

has simple infinities at a_1, a_2 ... a_n, b_i and ∞, simple zeros at b_1 ... b_{i-1}, b_{i+1} ... b_n, and finally a triple zero at 0.

Hence $\dfrac{1}{[x-x(b_i)]y}\dfrac{dx}{dz}$ has no infinity, and double zeros at the points $b_1 \dots b_{i-1}$, $b_{i+1} \dots b_n$, ∞, that is to say the equivalent of $2n$ simple zeros in the generating polygon; moreover, as already shown, it is of the form

$$\sum_i f(z_i)(\gamma_i z+\delta)^{-2}.$$

It therefore only remains to be shown that the constants C can be so chosen that

$$\sum_1^n C_p \theta(z, J_p),$$

which necessarily has a double zero at infinity, shall have $n-1$ other double zeros at the points $b_1, b_2 \dots b_{i-1}, b_{i+1} \dots b_n$.

This is immediately obvious; for, if $\theta(z, J)$ has a zero at z_0 in the generating polygon, it necessarily has a second at $-z_0$. Now if z_0 approaches one of the points b, a homologue of $-z_0$ will also approach b, and in the limit the two zeros will coincide and form a double zero. Hence, if the $n-1$ ratios of the constants are so determined that the expression in question vanishes at b_1, &c., it will necessarily have double zeros at these points.

Finally, then,

$$\frac{1}{[x-x(b_i)]y}\frac{dx}{dz} = \text{const.} \times \begin{vmatrix} \theta(z, J_1), & \theta(z, J_2), & \dots\dots & \theta(z, J_n) \\ \theta(b_1, J_1), & \dots\dots & & \theta(b_1, J_n) \\ \dots & \dots & \dots & \dots \\ \theta(b_{i-1}, J_1), & \dots\dots & & \theta(b_{i-1}, J_n) \\ \theta(b_{i+1}, J_1), & \dots\dots & & \theta(b_{i+1}, J_n) \\ \dots & \dots & \dots & \dots \\ \theta(b_n, J_1), & \dots\dots & & \theta(b_n, J_n) \end{vmatrix}.$$

Corresponding relations can obviously be obtained between the integrals of the second and third species and the corresponding z-functions.

2. *On the Prime Factors of an Automorphic Function.*

In his recent memoirs on hyper-elliptic and Abelian functions, Prof. F. Klein has introduced, as a new element, an expression to which he gives the name "*Primform.*"

He defines it as follows. Let $w_1, w_2, \ldots w_n$, be the n everywhere finite integrals connected with a given equation of deficiency n (or on a given $\overline{2n+1}$-ply connected Riemann's surface), and let $\phi_1, \phi_2, \ldots \phi_n$ be n finite (*i.e.*, not infinitesimal) quantities such that

$$\frac{dw_1}{\phi_1} = \frac{dw_2}{\phi_2} = \ldots = \frac{dw_n}{\phi_n}.$$

Then, if $d\omega$ is the common value of these fractions, and if $P^{r,\eta}_{x,y}$ is any integral of the third kind on the surface, the "*Primform*" $\Omega(x, y)$ is defined by the equation

$$\Omega(x, y) = \mathrm{Lt}._{dx=0, dy=0} \sqrt{d\omega_x d\omega_y \exp(-P^{x+dx,\, y+dy}_{x,y})},$$

x, y denoting any two points on the Riemann's surface.

$\Omega(x, y)$ vanishes only when the point x coincides with the point y, and does not become infinite for any position of the points.

It is the exact analogue for the multiply-connected surface of the form

$$x_1 y_2 - x_2 y_1$$

for a simply-connected surface, where x_1/x_2 is any function of position on such a simply connected surface, which takes every value once only; and just as any uniform function on the simply-connected surface can be expressed in the form

$$\prod_{y, y'} \frac{x_1 y_2 - x_2 y_1}{x_1 y_2' - x_2 y_1'},$$

so any uniform function on the multiply-connected surface can be expressed in the form

$$\prod_{y, y} \frac{\Omega(x, y)}{\Omega(x, y')}.$$

In his memoir of the theory of Abelian functions (*Math. Ann.*, Vol. XXXVI.), Prof. Klein refers to a paper by Herr Schottky in *Crelle's Journal*, Vol. CI., where a precisely similar expression is introduced in connexion with the theory of automorphic functions. From the point of view taken by Herr Schottky, the close parallel that exists between the functions he deals with and the analogous functions of position on a Riemann's surface, are lost sight of.

I propose then, here, to take Prof. Klein's definition, and from that directly to construct and investigate some of the properties of the

functions (and forms) on the z-plane that are equivalent to his *Primformen* and the derivable prime-functions on the surface.

It has been shown, p. 69 of my former paper, that the function

$$\chi_{a,b}(x, y) = \sum_i \log \frac{(z_i-a)(y_i-b)}{(z_i-b)(y_i-a)}$$

is the equivalent on the z-plane of the integral of the third kind on the Riemann's surface.

If the suffix refer to all substitutions except the identical one, this may be written

$$\exp \chi_{a,b}(z, y) = \frac{(z-a)(y-b)}{(z-b)(y-a)} \prod_i \frac{(z_i-a)(y_i-b)}{(z_i-b)(y_i-a)};$$

or, since $$\chi_{a,b}(z, y) = -\chi_{b,a}(z, y),$$

$$(z-b)(y-a) \exp\{-\chi_{b,a}(z, y)\} = (z-a)(y-b) \prod_i \frac{(z_i-a)(y_i-b)}{(z_i-b)(y_i-a)}.$$

If now a approaches y, and b approaches z, the right-hand side remains finite, and also in the limit becomes a perfect square, for it may be at once verified that cross ratios

$$\frac{(z_i-y)(y_i-z)}{(z_i-z)(y_i-y)} = \frac{(z_{-i}-y)(y_{-i}-z)}{(z_{-i}-z)(y_{-i}-y)}.$$

Hence the limiting form of the previous equation is

$$\mathrm{Lt}_{\substack{a=y \\ b=z}} \sqrt{(z-b)(a-y) \exp\{-\chi_{b,a}(z, y)\}} = (z-y) \prod_i \frac{(z_i-y)(y_i-z)}{(z_i-z)(y_i-y)},$$

where now, of each pair of inverse substitutions S and S^{-1}, only one is to be taken in the infinite product.

If now z_1/z_2 and y_1/y_2 be written for z and y, and the equation be multiplied all through by $y_2 z_2$, the left-hand side becomes

$$\mathrm{Lt}_{\substack{dz=0 \\ dy=0}} \sqrt{(z_1 dz_2 - z_2 dz_1)(y_2 dy_1 - y_1 dy_2) \exp\{-\chi_{z+dz, y+dy}(z, y)\}},$$

which agrees exactly with the expression used by Prof. Klein to define his *Primform*. The *Primform* on the z-plane may then be written

$$(zy) \prod_i \frac{(z_i y)(y_i z)}{(z_i z)(y_i y)},$$

where $$(zy) \equiv z_1 y_2 - z_2 y_1,$$

and the prime-function will be derived from this by dividing by y_2, z_2. It is to be remembered that in forming the infinite product, S and S^{-1} are not to be regarded as distinct substitutions. It is evidently immaterial for the present purposes whether the form or the function is dealt with, and a symbol $\Omega(z, y)$ will be taken for the function.

It is clear that $\Omega(z, y)$ has a single zero in each polygon, namely the homologue of y in that polygon; and that its only infinity is at infinity; also if z and y are interchanged the function changes sign.

To determine the relation between $\Omega(z_p, y)$ and $\Omega(z, y)$ it is necessary to return to the equation defining the function. This gives

$$\frac{\Omega(z_p, y)}{\Omega(z, y)} = \mathrm{Lt}_{\substack{a=y \\ b=z}} \sqrt{\frac{(z_p-b_p)(a-y)\exp\{-\chi_{b_p, a}(z_p, y)\}}{(z-b)(a-y)\exp\{-\chi_{b, a}(z, y)\}}}.$$

Now, from p. 69 of the former paper,

$$\chi_{b, a}(z_p, y) - \chi_{b, a}(z, y) = \phi_p(a) - \phi_p(b),$$

and

$$\chi_{b, a}(z, y) = \chi_{z, y}(b, a).$$

Hence

$$\begin{aligned} &\exp\{\chi_{b, a}(z, y) - \chi_{b_p, a}(z_p, y)\} \\ &\quad = \exp\{\chi_{b, a}(z, y) - \chi_{b, a}(z_p, y)\} \\ &\quad \times \exp\{\chi_{z_p, y}(b, a) - \chi_{z_p, y}(b_p, a)\} \\ &\quad = \exp\{\phi_p(b) - \phi_p(a) + \phi_p(z_p) - \phi_p(y)\}. \end{aligned}$$

Also,

$$\frac{z_p-b_p}{z-p} = \frac{1}{(\gamma_p z+\delta_p)(\gamma_p b+\delta_p)};$$

and therefore, writing $a = y$ and $b = z$,

$$\begin{aligned} \frac{\Omega(z_p, y)}{\Omega(z, y)} &= (\gamma_p z+\delta_p)^{-1} \exp \tfrac{1}{2}\{\phi_p(z_p) + \phi_p(z) - 2\phi_p(y)\} \\ &= (\gamma_p z+\delta_p)^{-1} \exp\left\{\phi_p(z) - \phi_p(y) + \frac{a_{pp}}{2}\right\}. \end{aligned}$$

If z_p be written for z in the product

$$\prod_1^m \frac{\Omega(z, y_r)}{\Omega(z, x_r)},$$

the function is reproduced multiplied by the factor

$$\exp \sum_1^m [\phi_p(x_r) - \phi_p(y_r)],$$

and if for n of the x's and y's suitably chosen homologues are written, this multiplier will become

$$\exp\left\{\sum_1^m\left[\phi_r(x_r)-\phi_p(y_r)\right]+\sum_1^n n_q a_{pq}\right\},$$

the quantities n_q being any arbitrarily assigned positive or negative integers; but when the x's and y's are the zeros and infinities of an automorphic function (p. 70 of the former paper) the integers n_q can be chosen so that the multipliers become

$$\exp\left[2m_p\pi i\right] \text{ or unity.}$$

Hence any automorphic function can be expressed as a product of prime-functions in the form

$$\prod_1^m \frac{\Omega(z, y_r)}{\Omega(z, x_r)}.$$

3. *On certain Sub-groups of the Symmetrical Group.*

If to the generating polygon of a symmetrical group be added the polygon into which it is transformed by one of the fundamental substitutions, say S_p, the new figure so formed is bounded by $4n-2$ circles, $2n-1$ of which are the inverses of the other $2n-1$ with respect to C_p, and which will therefore serve as the generating polygon of a new symmetric group. Moreover this new group is clearly a sub-group of the original one. Thus, with the notation of § 7 of my former paper, the fundamental substitutions of the new group are

$$\Sigma_1 = C_1'C_p, \quad \Sigma_2 = C_2'C_p, \ \ldots\ldots \ \Sigma_n = C_n'C_p,$$

$$\Sigma_{n+1} = C_1C_p, \ \ldots\ldots \ \Sigma_{2n+1} = C_nC_p,$$

Σ_{n+p} being the identical substitution.

Now the substitutions of the original group are

$$S_1 = C_1'C_0 \text{ (or } C_0C_1) \ \ldots\ldots \ S_n = C_n'C_0 \text{ (or } C_0C_n),$$

so that

$$\Sigma_1 = S_1S_p \ \ldots\ldots \ \Sigma_p = S_p^2, \ \ldots\ldots \ \Sigma_n = S_nS_p,$$

$$\Sigma_{n+1} = S_1^{-1}S_p, \ \ldots\ldots \ \Sigma_{2n+1} = S_n^{-1}S_p.$$

It is also clear that the substitutions of the original group are all given without repetition by the forms Σ and ΣS_p, where Σ represents the totality of the substitutions of the original group.

Using now accents to distinguish functions formed with the substitutions of the sub-group, it follows that

$$\theta(z, z_0) = \sum_i \left[\frac{1}{z - S_i\infty} - \frac{1}{z - S_i z_0}\right]$$

$$= \Sigma_i \left[\frac{1}{z - \Sigma_i\infty} + \frac{1}{z - \Sigma_i S_p\infty} - \frac{1}{z - \Sigma_i z_0} - \frac{1}{z - \Sigma_i S_p z_0}\right]$$

$$= \theta'(z, z_0) + \theta'(z, S_p z_0),$$

and
$$\theta(z, J_q) = \theta(z, z_0) - \theta(z, S_q z_0)$$
$$= \theta'(z, z_0) - \theta'(z, S_p S_q z_0) + \theta'(z, S_p z_0) - \theta'(z, S_q z_0)$$
$$= \theta'(z, J'_q) - \theta'(z, J'_{n+q-1});$$

while
$$\theta(z, J_p) = \theta'(z, J'_p).$$

Hence for the sub-group n linearly independent $\theta'(z, J')$ functions are identical with the n θ-functions of the original group; and this will involve certain simple relations between the corresponding quasi-periods of the ϕ' functions.

If now the original group is also fuchsian, so also is the sub-group. Suppose then that

$$x' = f(z)$$

is an unchanged function for the sub-group, that takes every value twice in its generating polygon. Then

$$f(z) + f(S_p z) \quad \text{and} \quad f(z) f(S_p z)$$

are unchanged functions for the original group, which take every value twice in its generating polygon, and therefore, x being the already used function for the original group, x' is the root of a quadratic equation whose roots are linear in x; or x is a rational function of the second degree of x'.

These considerations may be applied to a particular case of some interest, as follows. Suppose that for a symmetric fuchsian group derived from two fundamental substitutions (and leading therefore to hyperelliptic integrals of the first order), the three circles C_0, C_1, C_2 are symmetrical with respect to a fourth circle C, which must therefore cut one of them, say C_1, at right angles. The group is then a sub-group of that formed by an even number of inversions at the circles C, C_0, and C_1, and the two fundamental substitutions of this group being one hyperbolic and the other elliptic (of period two), the

corresponding algebraic equation is of deficiency 1, and the corresponding integrals elliptic. Hence the hyperelliptic integrals to which the group in question leads, are those which can be formed by a quadratic substitution from elliptic integrals. But these are of known form, and hence in this particular case a definite relation is obtained, without further calculation, between the nature of the group and the nature of the algebraic equation to which it leads. As a further verification, it may be very easily shown in this case, from the relation

$$\theta(z, J_p) = \theta'(z, J'_p),$$

proved on the last page, that

$$a_{12} = \tfrac{1}{2}a_{11},$$

which relation between the periods of the hyperelliptic integral involves the possibility of expressing it in terms of elliptic integrals by a quadratic substitution.

Note on Approximate Evolution. *By* H. W. Lloyd Tanner, M.A., F.R.A.S. Received June 6th, 1892. Read June 9th, 1892.

In a paper published in the *Proceedings* of the Society (Vol. xviii., pp. 171–178), Professor Hill has pointed out that the rule (Todhunter's *Algebra*, Art. 246) for contracting the process of finding the square and cube roots of a number, is incorrect in some cases. It is desirable to have a practical test for distinguishing the cases in which the rule is available from those in which it fails. Such a test is obtained by a slight modification of Todhunter's discussion (*loc. cit.*), which enables us also to state two limits between which the required root must lie.

Square Root.

It is convenient to take the decimal point in N, the number whose square root is to be found, so that the integral part of $\sqrt{N}$ may

sist of $2n+1$ figures. Of these the first $n+1$ are supposed to have been found, giving an approximate value, a, of the root. If x be the rest of the root, we have

$$N = (a+x)^2,$$

and, putting $$q = (N-a^2)/2a,$$

$$q = x + \frac{x^2}{2a}.$$

Hence, providing that $$\delta > \frac{x^2}{2a},$$

$$x+\delta > q > x;$$

or, what is the same thing,

$$q > x > q-\delta.$$

Now, from the meanings of a, x,

$$a \geqq 10^{2n}, \quad x < 10^n;$$

therefore $$\frac{x^2}{2a} < \frac{1}{2},$$

and we may take $\frac{1}{2}$ as a value of δ. It follows that

$$q > x > q-\tfrac{1}{2}.$$

Hence it is clear that, if q and $q-\frac{1}{2}$ have the same integral part, this is also the integral part of x. If, on the other hand, there be an integer between q and $q-\frac{1}{2}$, it is the nearest integer to x, but the integral part of x is not determined.

Since $q > x$, $q^2/2a > x^2/2a$. We may therefore take $\delta = q^2/2a$, a value which will usually, but not always, be less than the value assigned above. Hence

$$q > x > q - \frac{q^2}{2a}.$$

The limits for $\sqrt{N}$ thus obtained,

$$a+q > \sqrt{N} > a+q-\frac{q^2}{2a},$$

are the first two, or three, terms of the expansion of $(a^2+2aq)^{\frac{1}{2}}$.

Cube Root.

Suppose $\sqrt[3]{N}$ to have $2n+2$ figures in its integral part, of which $n+2$ have been determined, giving an approximate value a of the

root. Then, writing q for $(N-a^3)/3a^2$, and putting

$$N = (a+x)^3,$$

we have $$q = x + \frac{x^2}{a} + \frac{x^3}{3a^2};$$

therefore $$q > x > q-\delta,$$

if $$\delta > \frac{x^2}{a} + \frac{x^3}{3a^2}.$$

Now $$a \geqq 10^{2n+1}, \quad x < 10^n;$$

therefore $$\frac{x^2}{a} + \frac{x^3}{3a^2} < \frac{1}{10} + \frac{1}{3} \cdot \frac{1}{10^{n+2}}$$

$$< \frac{1}{10} + \frac{1}{3} \cdot \frac{1}{10^3},$$

so that we may take $$\delta = \cdot 100\dot{3}.$$

Hence, if the decimal part of q be less than $\cdot 100\dot{3}$, we cannot tell the integral part of x, but the nearest integer to x is known, being the integral part of q.

If the decimal part of q lie between $\cdot 600\dot{3}$ and $\cdot 5$ we cannot tell the nearest integer to x, but the integral part of x is the same as the integral part of q.

In all other cases we can tell both the greatest integer in x and its nearest integer.

The limits found for $\sqrt[3]{N}$, viz. $a+q$ and $a+q-\delta$, become respectively the first two and the first three terms of the expansion of $(a^3+3qa^2)^{\frac{1}{3}}$ by the binomial theorem, if q^2/a be put for δ. This is a proper value for δ; for, since

$$q > x + x^2/a$$

$$q^2/a > x^2/a + 2x^3/a^2 + x^4/a^3$$

$$> \frac{x^2}{a} + \frac{x^3}{3a^2}.$$

A Proof of the Exactness of Cayley's Number of Seminvariants of a given Type. *By* E. B. ELLIOTT. Received and read June 9th, 1892.

1. The general gradient, or rational integral homogeneous isobaric function, of order i and weight w in the n quantities $a_0, a_1, a_2, \ldots a_n$, is a sum of arbitrary multiples of $(w; i, n)$ products, where $(w; i, n)$ denotes the number of partitions of w into i or fewer numbers, none exceeding n.

Let us use the common notation Ω for the differential operator

$$a_0 \frac{d}{da_1} + 2a_1 \frac{d}{da_2} + 3a_2 \frac{d}{da_3} + \ldots + na_{n-1} \frac{d}{da_n}.$$

The result of operating with Ω on our supposed gradient of type w, i, n is a gradient of type $w-1, i, n$, *i.e.*, a sum of multiples of all or some of $(w-1; i, n)$ products. The vanishing of this, in case all the $(w-1; i, n)$ products occur, and have linearly independent coefficients, requires the satisfaction of $(w-1; i, n)$ conditions in the coefficients of the original gradient. In such a case then the number of linearly independent seminvariants—gradients annihilated by Ω—of type w, i, n is the difference $(w; i, n)-(w-1; i, n)$.

Should this difference be negative, the $(w-1; i, n)$ coefficients in the derived gradient of type $w-1, i, n$ cannot of course be all independent, being definite linear functions of a smaller number of original coefficients. If, however, the difference is zero or positive, it is not *a priori* clear whether all must be independent or not. Were there r linear relations connecting them, the number of linearly independent seminvariants of type w, i, n would be

$$(w; i, n)-(w-1; i, n)+r,$$

provided this be zero or positive, and not $(w; i, n)-(w-1; i, n)$,

The well-known facts are that, if $in \not< 2w$, the number of linearly independent seminvariants of type w, i, n is exactly

$$(w; i, n)-(w-1; i, n),$$

and that if on the other hand $in < 2w$, there are no seminvariants of

the type. These facts, stated much earlier by Cayley, were first demonstrated by Sylvester (*Phil. Mag.*, 1878), whose proof is reprinted in revised form in his "Lectures on Reciprocants" (*American Journal*, Vol. IX., Lecture 11).

Sylvester has also shown in the course of a second proof (Lecture 27) that, when $in-2w=-1$ in the general gradient ϕ of type w, i, n, no linear relation connects the coefficients in $\Omega\phi$, so that in this case also, as well as when $in-2w \not< 0$, the number $(w\,;\,i,\,n)-(w-1\,;\,i,\,n)$ of seminvariants is exact. There is no inconsistency with the general result in this, for the case is one in which $(w\,;\,i,\,n)=(w-1\,;\,i,\,n)$.

In the present paper, no attempt is made to improve on Sylvester's proof of the part of the theorem which asserts that when $in-2w<0$ there are no seminvariants of type w, i, n; nor is his quite simple demonstration of the fact reproduced. What is aimed at is so to use his fundamental lemmas as to obtain the other part of the theorem with no less facility, and thus to obviate the profound piece of reasoning by which he has established it.

In fact, that, if $in-2w \not< 0$ (or indeed $\not< -1$), the number of seminvariants of type w, i, n is precisely $(w\,;\,i,\,n)-(w-1\,;\,i,\,n)$, must follow if it can be shown that, when $in-2w \not< -1$, the result of operating with Ω on the most general gradient of type w, i, n is the production of the most general gradient of type $w-1, i, n$; and this must be the case if every gradient of type $w-1, i, n$, and in particular every separate product of that type, can be obtained by operating with Ω on some gradient or other of type w, i, n. Now that this is so will be found immediately to underlie Sylvester's lemmas.

For convenience, I will use $w+1$ instead of w.

2. Besides Ω take, as usual, the companion operator

$$O = na_1\frac{d}{da_0} + (n-1)\,a_2\frac{d}{da_1} + (n-3)\,a_3\frac{d}{da_2} + \ldots + a_n\frac{d}{da_{n-1}},$$

its designation O being Sylvester's.

What will be proved is that u, any gradient of type w, i, n, is the result of operating with Ω on the gradient of type $w+1, i, n$,

$$\frac{1}{\eta}\left\{O - \frac{1}{2(\eta+1)}O^2\Omega + \frac{1}{2.3\,(\eta+1)(\eta+2)}O^3\Omega^2 - \frac{1}{2.3.4(\eta+1)(\eta+2)(\eta+3)}O^4\Omega^3 + \ldots\right\}u,$$

where η denotes $in-2w$, and is supposed positive.

Corresponding to the characteristics i, w, η in u, we have i, $w-1$, $\eta+2$ in Ωu, and i, $w+1$, $\eta-2$ in Ou, and consequently i, $w-r+s-t$, $\eta+2(r-s+t)$ in $\Omega^r O^s \Omega^t u$.

Now Cayley and Sylvester have both used the readily proved identity

$$(\Omega O - O\Omega) u = \eta u,$$

and Sylvester has based his method of dealing with the present question on the more general equivalence

$$(\Omega O^r - O^r \Omega) u = r(\eta - r + 1) O^{r-1} u,$$

which is obtained by noticing that

$$\begin{aligned}(O^s \Omega O^{r-s} - O^{s+1} \Omega O^{r-s-1}) u &= O^s (\Omega O - O\Omega) O^{r-s-1} u \\ &= O^s (\eta - 2r + 2s + 2) O^{r-s-1} u \\ &= (\eta - 2r + 2s + 2) O^{r-1} u,\end{aligned}$$

and summing from $s = 0$ to $s = r-1$.

Now take in succession 1, 2, 3, 4, ... for r, and operate, not always on u, whose type is w, i, n, but in succession on u, Ωu, $\Omega^2 u$, $\Omega^3 u$, ..., remembering that the η characteristics of these gradients are respectively η, $\eta+2$, $\eta+4$, $\eta+6$, ... We thus find

$$\begin{aligned}(\Omega O - O\Omega) u &= \eta u, \\ (\Omega O^2 - O^2 \Omega) \Omega u &= 2(\eta+1) O \Omega u, \\ (\Omega O^3 - O^3 \Omega) \Omega^2 u &= 3(\eta+2) O^2 \Omega^2 u, \\ \&c. \qquad &\&c.;\end{aligned}$$

that is to say,

$$\begin{aligned}\eta u + O\Omega u &= \Omega O u, \\ 2(\eta+1) O\Omega u + O^2 \Omega^2 u &= \Omega O^2 \Omega u, \\ 3(\eta+2) O^2 \Omega^2 u + O^3 \Omega^3 u &= \Omega O^3 \Omega^2 u, \\ \&c. \qquad &\&c.\end{aligned}$$

After a time these equalities become mere identities of zeros, for $\Omega^{w+1} u$, at any rate, if no earlier $\Omega^r u$, vanishes, since its weight would be -1, an impossibility for any gradient or rational integral function.

The deduction is immediate, by addition of suitable multiples, alternately positive and negative, of the above equivalent pairs of gradients, that

$$u = \frac{1}{\eta}\,\Omega\left\{O - \frac{1}{2(\eta+1)}\,O^2\Omega + \frac{1}{2.3(\eta+1)(\eta+2)}\,O^3\Omega^2 - \ldots\right\}u,$$

the series being continued till its terms vanish, as must be the case from the $(w+2)^{\text{th}}$ onwards at any rate.

Thus a gradient of type $w+1, i, n$, equal to $\Omega^{-1}u$, where u is any gradient of type w, i, n, has been found, provided η or $in-2w$ is positive. As should be the case, there is failure to find any such gradient if η be zero or negative.

If, then, $in-2w > 0$, Ω, operating on the most general gradient of type $w+1, i, n$, can produce nothing less general than the most general gradient of type w, i, n. In this, replace w by $w-1$, and we deduce that, if $in-2w \not< -1$, the result of operating with Ω on the most general gradient of type w, i, n is the most general gradient of type $w-1, i, n$. Consequently, if $in-2w \not< -1$, the number of linearly independent seminvariants of order i, weight w, and extent n or less, is exactly $(w; i, n)-(w-1; i, n)$, and no greater number.

3. Our aim is now achieved; but it may not be out of place to append a few remarks in connexion with the method and the theorem.

It occurs, for instance, to consider what theorem replaces that of the determination of the form of $\Omega^{-1}u$ when η or $in-2w$ is zero for u. The answer clearly is that

$$\Omega\left\{O - \frac{1}{1.2}\,O^2\Omega + \frac{1}{1.2^2.3}\,O^3\Omega^2 - \frac{1}{1.2^2.3^2.4}\,O^4\Omega^3 + \ldots\right\}$$

annihilates u. Now we need not stop here, for Ω can annihilate no gradient of weight $w+1$ for which $in-2(w+1)$ is negative, as is here the case. Consequently

$$\left\{O - \frac{1}{1.2}\,O^2\Omega + \frac{1}{1.2^2.3}\,O^3\Omega^2 - \frac{1}{1.2^2.3^2.4}\,O^4\Omega^3 + \ldots\right\}u = 0,$$

u being any gradient for which $in-2w = 0$.

If $in-2w$ is negative ($=-\eta'$ say), we are in like manner led to two theorems: that

$$\Omega\left\{O+\frac{1}{2(\eta'-1)}O^2\Omega+\frac{1}{2.3(\eta'-1)(\eta'-2)}O^3\Omega^2+\ldots\right.$$
$$\left.\ldots+\frac{1}{2.3\ldots\eta'(\eta'-1)(\eta'-2)\ldots1}O^{\eta'}\Omega^{\eta'-1}\right\}u$$
$$=-\eta'u+\frac{1}{2.3\ldots\eta'(\eta'-1)(\eta'-2)\ldots1}O^{\eta'}\Omega^{\eta'}u,$$

and that

$$\Omega\left\{O^{\eta'+1}\Omega^{\eta'}-\frac{1}{1(\eta'+2)}O^{\eta'+2}\Omega^{\eta'+1}\right.$$
$$\left.+\frac{1}{1.2(\eta'+2)(\eta'+3)}O^{\eta'+3}\Omega^{\eta'+2}-\ldots\right\}u=0.$$

Of these two the first has apparently trifling interest. The second tells us that Ω annihilates a gradient for which $in-2w'$ is negative, which must accordingly itself vanish. Thus

$$O^{\eta'+1}\Omega^{\eta'}-\frac{1}{1.(\eta'+2)}O^{\eta'+2}\Omega^{\eta'+1}+\frac{1}{1.2(\eta'+2)(\eta'+3)}O^{\eta'+3}\Omega^{\eta'+2}-\ldots$$

is an annihilator of any gradient u for which $in-2w'$ ($=-\eta'$) is negative.

4. Returning to the case of $\eta>0$, other expressions for $\Omega^{-1}u$, not in general terminating, and consequently probably of little use, may be written down. For instance, since

$$\eta+O\Omega=\Omega O,$$
$$\eta O\Omega+(O\Omega)^2=\Omega O\,.\,O\Omega,$$
$$\eta(O\Omega)^2+(O\Omega)^3=\Omega O(O\Omega)^2,$$
$$\&c.,\qquad\&c.,$$

the operation being always on u, we get

$$\eta u=\Omega O\left\{1-\frac{1}{\eta}O\Omega+\frac{1}{\eta^2}(O\Omega)^2-\frac{1}{\eta^3}(O\Omega)^3+\ldots\right\}u,$$

the remainder after the r^{th} term on the right being

$$(-1)^r\frac{1}{\eta^{r-1}}(O\Omega)^r.$$

Should u then be such that for any value of r this vanishes, we should have the expression for $\Omega^{-1}u$,

$$\frac{1}{\eta}\, O\left\{1-\frac{1}{\eta}\, O\Omega+\frac{1}{\eta^2}(O\Omega)^2-\ldots+(-1)^{r-1}\frac{1}{\eta^{r-1}}(O\Omega)^{r-1}\right\}.$$

5. The application of § 2 obtained, by putting

$$a_n = a_0x^n + nb_1x^{n-1} + \frac{n(n-1)}{1.2}\, b_2x^{n-2}+\ldots+b_n = v, \text{ say},$$

$$a_{n-1} = a_0x^{n-1}+(n-1)\, b_1x^{n-2}+\frac{(n-1)(n-2)}{1.2}\, {}_2x^{n-3}+\ldots+b_{n-1} = \frac{1}{n}\frac{dv}{dx},$$

$$a_{n-2} = a_0x^{n-2}+(n-2)\, b_1x^{n-3}+\ldots+b_{n-2} = \frac{1}{n(n-1)}\frac{d^2v}{dx^2},$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$a_1 = a_0x+b_1 = \frac{1}{n(n-1)\ldots 2}\frac{d^{n-1}v}{dx^{n-1}},$$

$$a_0 = a_0 = \frac{1}{n(n-1)\ldots 2.1}\frac{d^nv}{dx^n},$$

deserves to be more widely appreciated than is perhaps the case. With this substitution, we have

$$\Omega = \frac{d}{dx},$$

and

$$\Omega^{-1} = \int^x,$$

and our conclusion, involved in the Cayley-Sylvester theorem itself, and not brought to light only by the present method, is that, if u be any product, of degree w in x, of i functions chosen from among v, a quantic in x of degree n, and its successive differential coefficients, or any linear function of such products, then, provided $in-2w>0$, $\int u\,dx$ is a rational integral function of $v, \frac{dv}{dx}, \ldots \frac{d^nv}{dx^n}$, with the addition of an arbitrary constant. It is, of course, clear that $\int u\,dx$ can be expressed in terms of v and its derived functions, but it is a fact hard to prove from first principles that it can be expressed integr[redacted]

in terms of them. We should expect a negative power of a_0 or $\frac{d^n v}{dx^n}$ to occur as a factor of the integral, and we have indirectly the conclusion that, were $(w+1\,;\ i,\ n) < (w\,;\ i,\ n)$, such must in fact be the case. A succession of integrations by parts of course gives such a form. Thus, returning to the Ω notation,

$$\begin{aligned}\Omega^{-1}u &= \frac{a_1}{a_0}u - \Omega^{-1}\left(\frac{a_1}{a_0}\Omega u\right)\\ &= \frac{a_1}{a_0}u - \frac{a_1^2}{2a_0^2}\Omega u + \Omega^{-1}\left(\frac{a_1^2}{2a_0^2}\Omega^2 u\right)\\ &= \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots\\ &\quad\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots\\ &= \frac{a_1}{a_0}u - \frac{a_1^2}{2a_0^2}\Omega u + \frac{a_1^3}{2.3a_0^3}\Omega^2 u - \frac{a_1^4}{2.3.4a_0^4}\Omega^3 u + \ldots,\end{aligned}$$

a series which must terminate. ... ...

6. Provided $in-2w > 0$, we have found an expression for $\Omega^{-1}u$ rational and integral in $a_0, a_1, a_2, \ldots a_n$, though it is not for a moment asserted that, knowing the fact that there is such an expression, there is no easier method for us to obtain the expression than by use of the formula. When $in-2w \not> 0$, we can use our knowledge to be sure that there is an expression for $\Omega^{-1}u$ or $\int u\,dx$ which has a_0^{-1} for a factor to a much lower power than that obtained by the integration-by-parts formula of the end of the last article, viz., to the r^{th} power, where r is 1 or the number next greater than $\frac{\eta'}{n}$, according as $in-2w = 0$ or $-\eta'$. For, on this supposition, we see that

$$a_0^r u$$

has n or $rn-\eta'$, in the one case or the other, for its η characteristic, which is accordingly positive. Whence $\Omega^{-1}(a_0^r u)$ is rational and integral ($= v$, say), and consequently

$$\Omega^{-1}u = \frac{1}{a_0^r}v.$$

[NOTE. *September 9th*, 1892.—Since communicating this paper I have noticed that the identity of § 2 may be given the more elegant form

$$\left\{1 - \frac{\Omega O}{1^2} + \frac{\Omega^2 O^2}{1^2.2^2} - \frac{\Omega^3 O^3}{1^2.2^2.3^2} + \ldots\right\} u = 0.]$$

Applications of a Theory of Permutations in Circular Procession to the Theory of Numbers.

By Major P. A. MacMahon.

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1. In the *Comptes Rendus* of the French Academy, 11th April, 1892, there appeared a note by M. E. Jablonshi, presented by M. C. Jordan, on the subject of permutations in circular procession.

Also, in the Appendix to his "Théorie des Nombres," t. 1, M. E. Lucas gives a short investigation, which he attributes to M. Moreau.

Both investigators reach the same result, which, in a slightly different notation, may be stated as follows:—

Let there be n things, of which α are of one kind, β of a second, γ of a third, and so forth,

$$\alpha+\beta+\gamma+\ldots = n;$$

let N be the greatest common factor of the numbers $\alpha, \beta, \gamma, \ldots$, and write

$$N = \frac{\alpha}{\alpha'} = \frac{\beta}{\beta'} = \frac{\gamma}{\gamma'} = \ldots;$$

further, let $LP(\alpha, \beta, \gamma, \ldots)$

and $CP(\alpha, \beta, \gamma, \ldots)$

denote respectively the numbers of permutations of the n things in linear and in circular procession, so that

$$LP(\alpha, \beta, \gamma, \ldots) = \frac{n!}{\alpha!\,\beta!\,\gamma!\ldots},$$

and $CP(\alpha, \beta, \gamma \ldots)$

is the number whose expression is sought.

If d be any divisor of N, including N itself and unity, and $\phi(d)$ the *totient* of d (Sylvester's nomenclature for the numbers of integers prime to and not superior to d), the result is

$$nCP(N\alpha', N\beta', N\gamma', \ldots) = \Sigma\phi(d)\, LP\left(\frac{N}{d}\alpha', \frac{N}{d}\beta', \frac{N}{d}\gamma', \ldots\right),$$

otherwise written

$$nCP(N\alpha', N\beta', N\gamma', \ldots) = \Sigma\phi(d)\,\frac{\left(\frac{n}{d}\right)!}{\left(\frac{N}{d}\alpha'\right)!\left(\frac{N}{d}\beta'\right)!\left(\frac{N}{d}\gamma'\right)!\ldots}.$$

This theorem involves results of interest and importance in the pure theory of numbers.

If n be prime, N is unity, and

$$nCP(\alpha, \beta, \gamma, \ldots) = \frac{n!}{\alpha!\,\beta!\,\gamma!\ldots},$$

and this is also true whenever, n being composite, $\alpha, \beta, \gamma, \ldots$ possess no common factor greater than unity. Hence

Theorem.—The multinomial coefficient

$$\frac{n!}{\alpha!\,\beta!\,\gamma!\ldots}$$

is divisible by n, provided that the numbers $\alpha, \beta, \gamma, \ldots$ constitute a prime assemblage.

The usual statement and proof in treatises is in reference to the divisibility by n, when n is prime, of every coefficient of the multinomial expansion except those attached to powers of single letters. The theorem above is more general and shows the nature of the quotient.

Next suppose $N = n = \alpha = N\alpha'$;

we find $nCP(n) = \Sigma\phi(d)$,

and manifestly $CP(n) = 1$;

therefore $\Sigma\phi(d) = n$,

where d is any divisor of n, including n itself and unity; the well-known theorem due to Gauss.

The complete theorem may be viewed as a generalization of that of Gauss. The expression

$$\Sigma\phi(d)\,\frac{\left(\frac{n}{d}\right)!}{\left(\frac{N}{d}\alpha'\right)!\left(\frac{N}{d}\beta'\right)!\left(\frac{N}{d}\gamma'\right)!\ldots}$$

gives, for every partition of n, a linear function of the totients of the

divisors of n which is divisible by n. Moreover the quotient is

$$CP\,(N\alpha', N\beta', N\gamma', \ldots).$$

It may be noted that, when N is a prime,

$$nCP(N\alpha', N\beta', N\gamma' \ldots) = \frac{n!}{(N\alpha')!\,(N\beta')!\,(N\gamma')! \ldots} + (N-1)\frac{\left(\frac{n}{N}\right)!}{\alpha'!\,\beta'!\,\gamma'! \ldots},$$

and then $(\alpha'\beta'\gamma' \ldots)$ is any partition of $\frac{n}{N}$.

2. The next problem is with reference to the permutations of p different objects, n at a time, when repetitions of objects are permissible.

The number of line permutations may be denoted by $RLP\,(p, n)$, and the number of circular permutations by $RCP\,(p, n)$.

The value of $RLP\,(p, n)$ is known to be p^n.

In a circular permutation there are n objects which may be all similar, or they may be of one, two, &c. ... p different kinds.

A permutation which involves α_1 objects of one kind, α_2 of a second, α_3 of a third, &c., may be said to be of type

$$(\alpha_1 \alpha_2 \ldots \alpha_p),$$

where $$\alpha_1 + \alpha_2 + \ldots + \alpha_p = n,$$

and 1, 2, ... or $p-1$ of the quantities α may be zero.

In order to place in evidence the existence of equalities amongst the quantities α, it is convenient to consider the general type

$$(\alpha_1^{\kappa_1}\, \alpha_2^{\kappa_2}\, \alpha_3^{\kappa_3} \ldots),$$

where $$\kappa_1 + \kappa_2 + \kappa_3 + \ldots = p,$$

and $$\kappa_1\alpha_1 + \kappa_2\alpha_2 + \kappa_3\alpha_3 + \ldots = n.$$

We have to find the number of different combinations of objects which come under a given type.

Were the type $(\alpha_1, \alpha_2, \ldots \alpha_p)$, with the quantities α all different, the number of different combinations would be clearly $p!$.

Hence, for the general type, the number of different combinations of the objects must be

$$\frac{p!}{\kappa_1!\,\kappa_2!\,\kappa_3! \ldots}.$$

Ex. gr., if a permutation involves $n-2$ objects of one kind, one of a second, and one of a third, the type is

$$(n-2,\ 1^2,\ 0^{p-3}),$$

and the number of different combinations of this type is

$$\frac{p!}{1!\,2!\,(p-3)!}.$$

Denote by $CP(a_1^{\kappa_1}, a_2^{\kappa_2}, a_3^{\kappa_3}, \ldots)$ the number of circular permutations of a set of objects of the type

$$(a_1^{\kappa_1} a_2^{\kappa_2} a_3^{\kappa_3} \ldots);$$

then
$$RCP(p, n) = \Sigma \frac{p!}{\kappa_1!\,\kappa_2!\,\kappa_3! \ldots} CP(a_1^{\kappa_1} a_2^{\kappa_2} a_3^{\kappa_3} \ldots).$$

But
$$nCP(a_1^{\kappa_1} a_2^{\kappa_2} a_3^{\kappa_3} \ldots) = \Sigma\phi(d)\, LP\left\{\left(\frac{a_1}{d}\right)^{\kappa_1}\left(\frac{a_2}{d}\right)^{\kappa_2}\left(\frac{a_3}{d}\right)^{\kappa_3}\ldots\right\},$$

by the previous theorem, d denoting a divisor of the highest common factor of the numbers a.

We can now express $RCP(p, n)$ as a linear function of the expressions LP. We find that the coefficient of $\frac{1}{n}\phi(d)$ is

$$\Sigma \frac{p!}{\kappa_1!\,\kappa_2!\,\kappa_3! \ldots} LP\left\{\left(\frac{a_1}{d}\right)^{\kappa_1}\left(\frac{a_2}{d}\right)^{\kappa_2}\left(\frac{a_3}{d}\right)^{\kappa_3}\ldots\right\},$$

and this series is manifestly the value of

$$RLP\left(p, \frac{n}{d}\right),$$

arrived at by summation in regard to types. Hence

$$RCP(p, n) = \frac{1}{n}\Sigma\phi(d)\, RLP\left(p, \frac{n}{d}\right),$$

or
$$RCP(p, n) = \frac{1}{n}\Sigma\phi(d)\, p^{n/d},$$

the summation having reference to every divisor d of the number n.

Theorem. — The number of circular permutations of p different objects n together, repetitions permissible, is equal to

$$\frac{1}{n}\Sigma\phi(d)\, p^{n/d},$$

where $\phi(d)$ represents the totient of d, and the summation is for all divisors d of the number n.

We have here another extension of Gauss's theorem concerning totients. Corresponding to every integer p, we have a linear function of the number $\phi(d)$, d a divisor of n, which is divisible by n, and the theorem shows the nature of the quotient.

Theorem.—If n and p be any positive integers, and d a divisor of n, the sum

$$\Sigma \phi(d)\, p^{n/d} \equiv 0 \mod n,$$

and the quotient is equal to the number of circular permutations of p different things n together, repetitions permitted.

Gauss's theorem is given by p equal to unity.

The theorem is also a generalization of Fermat's theorem concerning the divisibility of $p^{n-1}-1$ by n, when n is a prime and p prime to n.

For, since $\Sigma \phi(d)\, p^{n/d} \equiv 0 \mod n$,

if n be prime, $p^n + (n-1)\, p \equiv 0 \mod n$,

or $p^{n-1} - 1 \equiv 0 \mod n$,

if p be prime to n, and

$$\frac{1}{n}(p^{n-1}-1) = \frac{1}{p} RCP(p, n) - 1,$$

showing the connexion between the quotient in Fermat's theorem and the number $RCP(p, n)$.

The usual extension of Fermat's theorem asserts the congruence

$$p^{\phi(N)} - 1 \equiv 0 \mod N,$$

when p is prime to N.

To deduce this from the general formula, suppose N to involve the prime m to the power μ, and put

$$N = N' m^{\mu}.$$

Consider the permutations in circular procession of $p^{\phi(N')}$ objects m^{μ} together, repetitions allowed.

By the formula established above

$$m^{\mu} RCP\left\{p^{\phi(N')}, m^{\mu}\right\}$$

$$= p^{\phi(N') m^{\mu}} + (m-1)\, p^{\phi(N') m^{\mu-1}} + \ldots + (m^{\mu} - m^{\mu-1})\, p^{\phi(N')}$$

$$= p^{\phi(N') m^{\mu-1}} \left\{p^{\phi(N' m^{\mu})} - 1\right\} + m p^{\phi(N') m^{\mu-2}} \left\{p^{\phi(N' m^{\mu-1})} - 1\right\} + \ldots + m^{\mu} p^{\phi(N')};$$

or
$$RCP\left\{p^{\phi(N')}, m^{\mu}\right\}$$
$$= p^{\phi(N')} + \frac{p^{\phi(N')}}{m}\left\{p^{\phi(N'm)}-1\right\} + \frac{p^{\phi(N')m}}{m^2}\left\{p^{\phi(N'm^2)}-1\right\}$$
$$+ \ldots + \frac{p^{\phi(N')m^{\mu-1}}}{m^{\mu}}\left\{p^{\phi(N)}-1\right\},$$

where the number of terms on the dexter is $\mu+1$.

Giving μ successive integer values, we establish the congruence

$$p^{\phi(N)}-1 \equiv 0 \mod m^{\mu},$$

and thence
$$p^{\phi(N)}-1 \equiv 0 \mod N.$$

To find the quotient of $p^{\phi(N)}-1$ by m^{μ}, write

$$RCP\left\{p^{\phi(N')}, m^{\mu}\right\} = R_{\mu},$$
$$\frac{1}{m^{\mu}}\left\{p^{\phi(Nm^{\mu})}-1\right\} = \vartheta_{\mu},$$
$$p^{\phi(N')} = P,$$

so that
$$R_{\mu} = P + P\vartheta_1 + P^m\vartheta_2 + \ldots + P^{m^{\mu-1}}\vartheta_{\mu},$$

and hence
$$\vartheta_{\mu} = P^{-m^{\mu-1}}(R_{\mu}-R_{\mu-1}),$$

or
$$\frac{1}{m^{\mu}}\left\{p^{\phi(N)}-1\right\}$$
$$= p^{-\phi(N/m^{\mu})m^{\mu-1}}\left[RCP\left\{p^{\phi(N/m^{\mu})}, m^{\mu}\right\} - RCP\left\{p^{\phi(N/m^{\mu})}, m^{\mu-1}\right\}\right]$$
$$= \vartheta_{\mu}.$$

Let now
$$N = m_1^{\mu_1} m_2^{\mu_2} \ldots m_s^{\mu_s},$$

then
$$\frac{\left\{p^{\phi(N)}-1\right\}^s}{N} = \vartheta_{\mu_1}\vartheta_{\mu_2}\ldots\vartheta_{\mu_s},$$

leading to
$$\frac{p^{\phi(N)}-1}{N} = (\vartheta_{\mu_1}\vartheta_{\mu_2}\ldots\vartheta_{\mu_s})^{1/s} N^{-(s-1)/s},$$

giving the complete quotient.

Ex. gr., $N=15$, $m_1=3$, $m_2=5$, $\mu_1=\mu_2=1$, $s=2$,
$$\vartheta_{\mu_1} = \tfrac{1}{4}RCP(4,5)-1 = \tfrac{1}{20}(4^5+4^2)-1 = 51,$$
$$\vartheta_{\mu_2} = \tfrac{1}{16}RCP(16,3)-1 = \tfrac{1}{48}(16^3+2.16)-1 = 85;$$

therefore $$(\vartheta_{\mu_1}\vartheta_{\mu_2})^{\frac{1}{2}} N^{-\frac{1}{2}} = \sqrt{51}\,\sqrt{85}\,\frac{1}{\sqrt{15}} = 17,$$

which is right.

3. To extend Fermat's theorem in another direction, expand the right-hand side of the formula

$$nRCP(p, n) = \Sigma\phi(d)\, p^{n/d},$$

according to powers of the prime factors of n.

It is convenient to adopt a symbolic notation by which

$$p^{[a]} = p^{a-1},$$

$$p^{[a]}p^{[b]}p^{[c]}\ldots = p^{[abc\ldots]} = p^{abc\ldots-1}.$$

As a simple case, put $n = m_1m_2m_3$, and we have

$$p^{m_1m_2m_3} + (m_1-1)\,p^{m_2m_3} + (m_2-1)\,p^{m_1m_3} + (m_3-1)\,p^{m_1m_2}$$
$$+ (m_1-1)(m_2-1)\,p^{m_3} + (m_2-1)(m_3-1)\,p^{m_1} + (m_3-1)(m_1-1)\,p^{m_2}$$
$$+ (m_1-1)(m_2-1)(m_3-1)\,p \equiv 0 \bmod m_1m_2m_3;$$

p being prime to n, a rearrangement gives

$$(p^{[m_1]}-1)(p^{[m_2]}-1)(p^{[m_3]}-1)$$
$$+ m_1(p^{[m_2]}-1)(p^{[m_3]}-1) + m_2(p^{[m_3]}-1)(p^{[m_1]}-1)$$
$$+ m_3(p^{[m_1]}-1)(p^{[m_2]}-1)$$
$$+ m_2m_3(p^{[m_1]}-1) + m_3m_1(p^{[m_2]}-1) + m_1m_2(p^{[m_3]}-1)$$
$$+ m_1m_2m_3 \equiv 0 \bmod m_1m_2m_3.$$

Now, by Fermat's theorem,

$$p^{[m_1]}-1 \equiv 0 \bmod m_1,$$

and, taking n the product of two primes, a formula similar to the above shows the congruence

$$(p^{[m_2]}-1)(p^{[m_3]}-1) \equiv 0 \bmod m_2m_3.$$

Hence the congruence

$$(p^{[m_1]}-1)(p^{[m_2]}-1)(p^{[m_3]}-1) \equiv 0 \bmod m_1m_2m_3,$$

which is equivalent to

$$p^{m_1m_2m_3-1} - p^{m_2m_3-1} - p^{m_3m_1-1} - p^{m_1m_2-1}$$
$$+ p^{m_1-1} + p^{m_2-1} + p^{m_3-1} - 1 \equiv 0 \bmod m_1m_2m_3.$$

To generalize this congruence write

$$\Phi_p(m_1^{\mu_1} m_2^{\mu_2} \dots m_s^{\mu_s}) = p^{\left[m_1^{\mu_1-1}\right]} (p^{[m_1]}-1) \dots p^{\left[m_s^{\mu_s-1}\right]} (p^{[m_s]}-1),$$

according to the law of the totient formula

$$\phi(m_1^{\mu_1} m_2^{\mu_2} \dots m_s^{\mu_s}) = m_1^{\mu_1-1}(m_1-1)\, m_2^{\mu_2-1}(m_2-1) \dots m_s^{\mu_s-1}(m_s-1).$$

The above-established congruence is then written

$$\Phi_p(m_1 m_2 m_3) \equiv 0 \bmod m_1 m_2 m_3.$$

We can establish the congruence

$$\Phi_p(m_1^{\mu_1} m_2^{\mu_2} \dots m_s^{\mu_s}) \equiv 0 \bmod m_1^{\mu_1} m_2^{\mu_2} \dots m_s^{\mu_s},$$

p being prime to $m_1^{\mu_1} m_2^{\mu_2} \dots m_s^{\mu_s}$.

If $m_1^{a_1} m_2^{a_2} \dots m_s^{a_s}$ and $m_1^{b_1} m_2^{b_2} \dots m_s^{b_s}$ be conjugate divisors of $m_1^{\mu_1} m_2^{\mu_2} \dots m_s^{\mu_s}$, we may write

$$n \Sigma \phi(d)\, p^{n/d} = \Sigma \phi(m_1^{a_1} m_2^{a_2} \dots m_s^{a_s})\, p^{m_1^{b_1} m_2^{b_2} \dots m_s^{b_s}},$$

where

$$\phi(m_1^{a_1} m_2^{a_2} \dots m_s^{a_s}) = (m_1^{a_1} - m_1^{a_1-1})(m_2^{a_2} - m_2^{a_2-1}) \dots (m_s^{a_s} - m_s^{a_s-1});$$

therefore $\Sigma \phi(m_1^{a_1} m_2^{a_2} \dots m_s^{a_s})\, p^{\left[m_1^{b_1} m_2^{b_2} \dots m_s^{b_s}\right]} \equiv 0 \bmod n.$

On the left-hand side, expansion of

$$\phi(m_1^{a_1} m_2^{a_2} \dots m_s^{a_s})$$

shows that the whole coefficient of $m_1^{a_1} m_2^{a_2} \dots m_s^{a_s}$ is

$$p^{\left[m_1^{b_1} m_2^{b_2} \dots m_s^{b_s}\right]} - p^{\left[m_1^{b_1-1} m_2^{b_2} \dots m_s^{b_s}\right]} - p^{\left[m_1^{b_1} m_2^{b_2-1} \dots m_s^{b_s}\right]} - \dots$$
$$+ p^{\left[m_1^{b_1-1} m_2^{b_2-1} \dots m_s^{b_s}\right]} + p^{\left[m_1^{b_1} m_2^{b_2-1} m_3^{b_3-1} \dots m_s^{b_s}\right]} + \dots$$
$$- \dots,$$

which is

$$\left(p^{\left[m_1^{b_1}\right]} - p^{\left[m_1^{b_1-1}\right]}\right)\left(p^{\left[m_2^{b_2}\right]} - p^{\left[m_1^{b_2-1}\right]}\right) \dots \left(p^{\left[m_s^{b_s}\right]} - p^{\left[m_s^{b_s-1}\right]}\right),$$

that is, $\Phi_p(m_1^{b_1} m_2^{b_2} \dots m_s^{b_s})$.

Hence $\Sigma m_1^{a_1} m_2^{a_2} \dots m_s^{a_s} \Phi_p (m_1^{b_1} m_2^{b_2} \dots m_s^{b_s}) \equiv 0 \mod n$;

or, more briefly,

$$\Sigma d \Phi_p \left(\frac{n}{d}\right) \equiv 0 \mod n,$$

which is simply another form of the congruence

$$\Sigma \phi (d) p^{n/d} \equiv 0 \mod n.$$

The congruence $\Sigma d \Phi_p \left(\frac{n}{d}\right) \equiv 0 \mod n$

may be written $\Phi_p (n) + \Sigma' d \Phi_p \left(\frac{n}{d}\right) \equiv 0 \mod n,$

wherein the summation sign Σ' has reference to every divisor of n *excluding* unity.

Assuming the truth of the congruence

$$\Phi_p \left(\frac{n}{d}\right) \equiv 0 \mod \frac{n}{d},$$

for every divisor d of n which exceeds unity, manifestly

$$\Sigma' d \Phi_p \left(\frac{n}{d}\right) \equiv 0 \mod n,$$

and hence $\Phi_p (n) \equiv 0 \mod n$;

therefore by induction universally

$$\Phi_p (n) \equiv 0 \mod n;$$

or, otherwise, $\Phi_p (m_1^{\mu_1} m_2^{\mu_2} \dots m_s^{\mu_s}) \equiv 0 \mod m_1^{\mu_1} m_2^{\mu_2} \dots m_s^{\mu_s}$;

and, since $\Phi_p (m_1^{\mu_1} m_2^{\mu_2} \dots m_s^{\mu_s})$ contains a factor $p^{N/(m_1 m_2 \dots m_s)-1}$, we have finally

$$p^{1-N/(m_1 m_2 \dots m_s)} \Phi_p (m_1^{\mu_1} m_2^{\mu_2} \dots m_s^{\mu_s}) \equiv 0 \mod m_1^{\mu_1} m_2^{\mu_2} \dots m_s^{\mu_s}.$$

Ex. gr., Take $n = m_1^3 m_2,$

$$p^{1-m_1^2} (p^{[m_1^2]} - p^{[m_1^2]}) (p^{[m_2]} - 1) \equiv 0 \mod m_1^3 m_2,$$

or $$p^{1-m_1^2} (p^{m_1^2 m_2 - 1} - p^{m_1^2 - 1} - p^{m_1^2 m_2 - 1} + p^{m_1^2 - 1}) \equiv 0 \mod m_1^3 m_2,$$

or $$p^{m_1^2 m_2 - m_1^2} - p^{m_1^2 - m_1^2} - p^{m_1^2 m_2 - m_1^2} + 1 \equiv 0 \mod m_1^3 m_2.$$

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The following is the list of mathematical books bequeathed by the late Dr. Thomas Archer Hirst, F.R.S., &c., to the President and Council, for the time being, of the Society* :—

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*On Selective and Metallic Reflection, pp. 4–18.
*On the Theory of Elastic Wires, pp. 105–127.
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*A Proof of the Exactness of Cayley's Number of Seminvariants of a Given Type, pp. 298–304.

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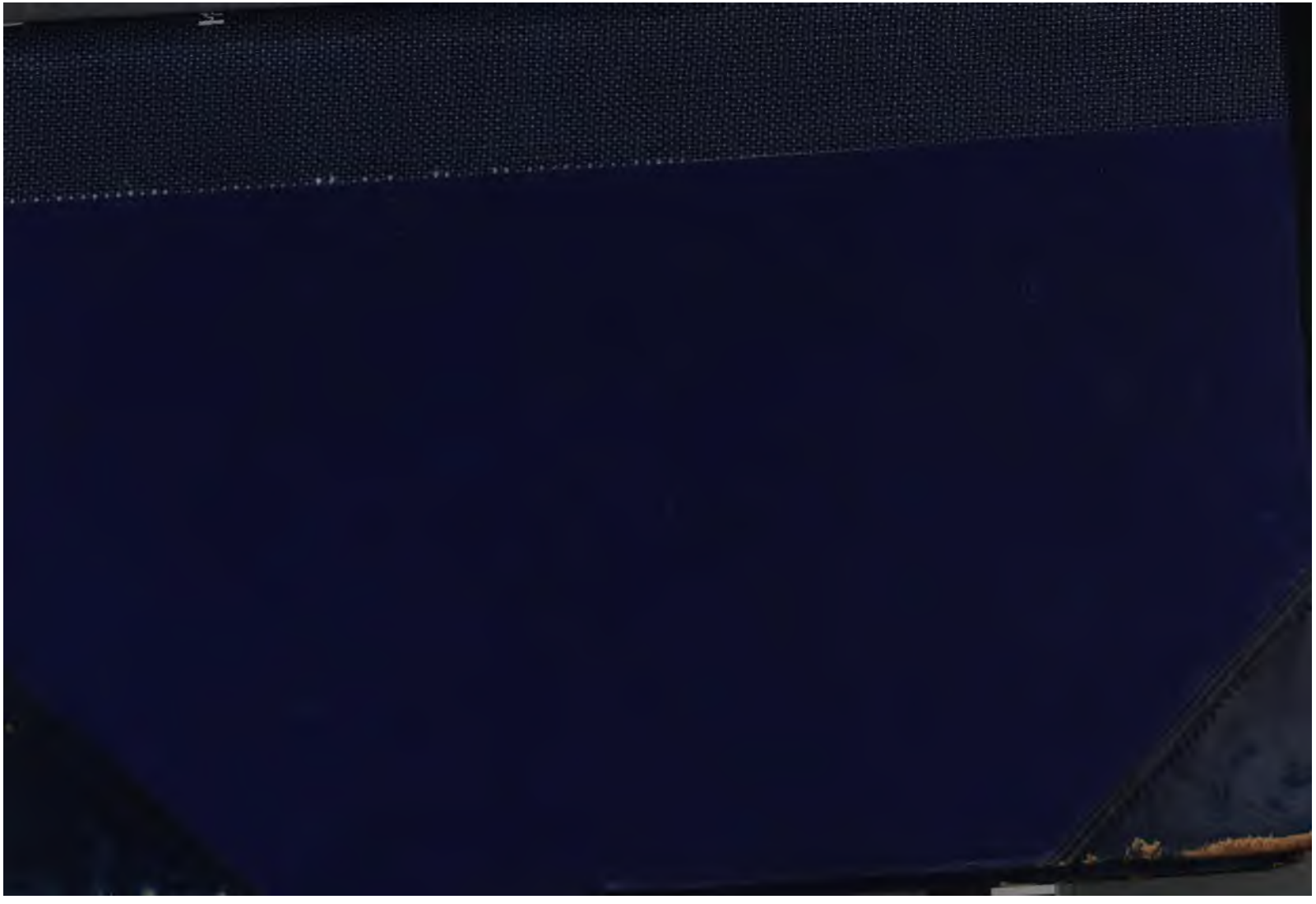
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